CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 9 Supplement

1. Introduction

The following material has turned out to be quite technical. I would say that the "take-home" messages of this document are the following.

- The geodesic exponential map is a *local diffeomorphism*, meaning that $\exp_p$ maps an open disk containing the origin in T_pS continuous, bijectively and with continuous inverse onto an open neighbourhood of p in S.
- The exponential map is thus a parametrization of a portion of S. How "good" is this parametrization? It's not so good that it maps the inner product in the parameter domain to the inner product of Euclidean space (that would be asking a lot!). Rather, it preserves these inner products to a certain extent (the Gauss Lemma).
- Geodesics locally minimize length in the sense that if $p, q \in S$ are sufficiently close together, then there exists a geodesic from p to q and it is the unique shortest curve between them.

2. The Exponential Map Is A Local Diffeomorphism

Let S be a regular surface and $p \in S$. We have defined the exponential map $\exp_p : T_pS \to S$ as "the map from initial condition to solution" of the geodesic equation. In other words, $\exp_p(X) = \gamma(1)$ where γ is the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = X$. Furthermore, by the "uniqueness" part of the Existence and Uniqueness Theorem of Second-Order ODEs, we can show that $\exp_p(tX) = \gamma(t)$.

A very simple observation with a significant consequence is that $[D\exp_p]_0$ is the identity map. But this is hard to see through all the notation. Let's verify this result first, and then we'll state its consequence in the next paragraph. Recall that by definition, $[D\exp_p]_0(V) := \frac{d}{dt}\exp_p(V(t))\big|_{t=0}$ where $V(t)$ is any curve in T_pS such that $V(0) = 0$ and $\dot{V}(0) = V$. Since T_pS is a vector space where multiplication by a scalar makes sense, we can choose $V(t) := tV$. Thus

$$[D\exp_p]_0(V) = \frac{d}{dt}\exp_p(tV)\bigg|_{t=0} = V \tag{1}$$

again by definition of the exponential map as the unique geodesic $\gamma(t) := \exp_p(tV)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = V$. But (**??**) states exactly that $[D\exp_p]_0 = id$.

The consequence of this result follows from the inverse function theorem, which states that if the differential of a map is invertible then the map itself is a *local diffeomorphism.* Thus $\exp_p : T_pS \to S$ is a local diffeomorphism. This means that there is a radius R so that the ball of radius R about the origin in T_pS maps diffeomorphically (smoothly, bijectively, with smooth inverse) onto a subset $\mathcal{B} \subseteq S$ containing p. We call $\mathcal{B}$ a *normal neighbourhood* of p.

Another way to say this is: we can use $\exp_p : B_R(0) \subseteq T_pS \to \mathcal{B} \subseteq S$ to parametrize a portion of S, namely the normal neighbourhood $\mathcal{B}$, from a portion of the tangent plane T_pS, namely $B_R(0)$.

3. The Gauss Lemma

We can ask what happens to inner products of vectors under $[D\exp_p]$. The result is known as the Gauss Lemma. The set-up is best visualized by picture — see Figure **??**.

Lemma. *Let $v, w \in T_v(T_pS)$. Then $\big\langle [D\exp_p]_v(v), [D\exp_p]_v(w) \big\rangle = \langle v, w \rangle$.*

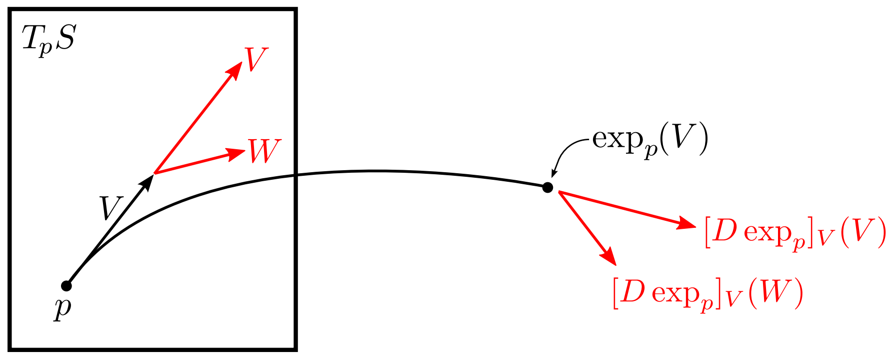

Figure 1: The quantities appearing in the statement of the Gauss Lemma. The white page represents the surface S, the rectangle represents T_pS and the curve represents the geodesic $t \mapsto \exp_p(tv)$.

Proof. Since $[D\exp_p]_v$ is a linear map, it is sufficient to split $w = w^{\parallel} + w^{\perp}$ into a component $w^{\parallel}$ parallel to v and a component $w^{\perp}$ perpendicular to v, and then to prove the Gauss Lemma separately for $w^{\parallel}$ and $w^{\perp}$. Moreover, it is sufficient to take $w^{\parallel} = v$. In summary, thanks to linearity, we have to prove only two things: $\big\langle [D\exp_p]_v(v), [D\exp_p]_v(v)\big\rangle = \langle v, v\rangle$ and $\big\langle [D\exp_p]_v(v), [D\exp_p]_v(w^{\perp})\big\rangle = 0$. Also, we're going to use the definition of the differential of a map F as $[DF]_x(V) := \frac{d}{d\varepsilon}F(c(\varepsilon))\big|_{\varepsilon=0}$ where c is any curve satisfying $c(0) = x$ and $\dot{c}(0) = X$.

For the first calculation, we use the definition of the differential of a map to write $[D\exp_p]_v(v) = \frac{d}{ds}\exp_p((1+s)v)\big|_{s=0}$ which equals $\frac{d}{dt}\exp_p(tv)\big|_{t=1}$ just by changing variables $t = 1+s$. Therefore

$$\big\langle [D\exp_p]_v(v), [D\exp_p]_v(v)\big\rangle = \left\langle \frac{d}{dt}\exp_p(tv), \frac{d}{dt}\exp_p(tv)\right\rangle\bigg|_{t=1} = \langle \dot\gamma(1), \dot\gamma(1)\rangle$$

where $\gamma(t) := \exp_p(tv)$ is the geodesic through p with $\dot\gamma(0) = v$. We evaluate this inner product indirectly as follows. Note that $\frac{d}{dt}\langle\dot\gamma(t), \dot\gamma(t)\rangle = 2\langle\ddot\gamma(t), \dot\gamma(t)\rangle = 0$ for all t. This is because γ satisfies the geodesic equation (the inner product of $\ddot\gamma$ — which equals $\vec{k}_\gamma$ because γ is parametrized by a constant multiple of arc-length by definition — with any tangential vector vanishes!) Therefore $\langle\dot\gamma(t), \dot\gamma(t)\rangle$ is constant. This implies that $\langle v, v\rangle = \langle\dot\gamma(0), \dot\gamma(0)\rangle = \langle\dot\gamma(1), \dot\gamma(1)\rangle$ as required.

Next we tackle the second calculation. The principle is the same, but we have to be a bit more clever. Define the function $f(s,t) := \exp_p(tv(s))$ where $v(s)$ is a curve in T_pS satisfying $v(0) = v$ and $\dot v(0) = w$ where $v \perp w$. Moreover, since any curve will do, and since all we really care about is the $v \perp w$, we can choose $v(s)$ so that $\|v(s)\|$ is constant. (This works because $\|v(s)\| = const$ implies $\langle v, w\rangle = \langle v, \frac{dv}{ds}\rangle = \frac{1}{2}\frac{d}{ds}\|v(s)\|^2 = 0$.) See Figure **??**.

By definition of the exponential of a map (think about this!), we can say

$$\big\langle [D\exp_p]_v(v), [D\exp_p]_v(w^{\perp})\big\rangle = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right\rangle\bigg|_{\substack{t=1\\ s=0}}$$

Since we don't know what this is, we proceed indirectly as above:

$$\begin{aligned}
\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right\rangle &= \left\langle \frac{\partial^2 f}{\partial t^2}, \frac{\partial f}{\partial s}\right\rangle + \left\langle \frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial t\partial s}\right\rangle \\
&= \left\langle \frac{\partial^2 f}{\partial t^2}, \frac{\partial f}{\partial s}\right\rangle + \left\langle \frac{\partial f}{\partial t}, \frac{\partial^2 f}{\partial s\partial t}\right\rangle \\
&= \left\langle \frac{\partial^2 f}{\partial t^2}, \frac{\partial f}{\partial s}\right\rangle + \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle
\end{aligned}$$

$$= \left\langle \frac{\partial^2 f}{\partial t^2}, \frac{\partial f}{\partial s} \right\rangle + \frac{1}{2}\frac{\partial}{\partial s}\|v(s)\|^2$$
$$= 0$$

for two reasons: the first term vanishes because the geodesic equation for the curve $t \mapsto \exp_p(tv(s))$ says that its second derivative is perpendicular to any tangent vector of S, in particular to $\frac{\partial f}{\partial s}$; the second term vanishes because $\|v(s)\|$ is constant. Consequently, $\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle$ is constant in t and so

$$\langle [D\exp_p]_v(v), [D\exp_p]_v(w^\perp) \rangle = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle \Big|_{\substack{t=1\\ s=0}} = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle \Big|_{\substack{t=0\\ s=0}} = 0$$

as required. The Gauss Lemma is thus proved! □

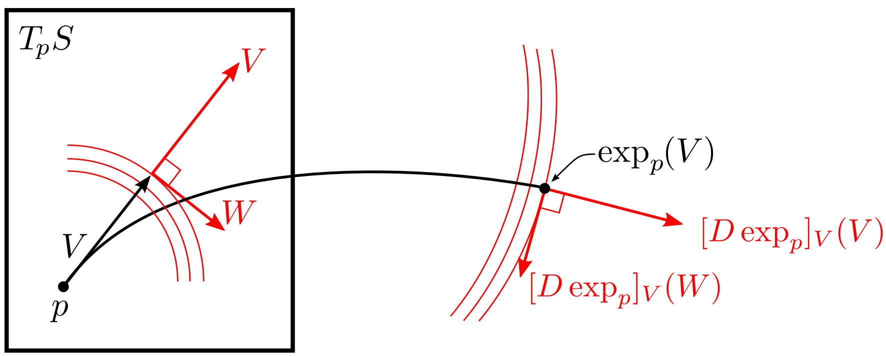

Figure 2: The red lines shown in T_pS are the curves $s \mapsto tV(s)$ for three values of t, including $t = 1$. W.l.o.g. we can make them concentric circles. Their images under $\exp_p$ are shown passing through $\exp_p(V)$ as well.

4. Geodesics Locally Minimize Length

We can now prove the main theorem of the lecture.

Theorem 1. *Geodesics locally minimize length: if γ is a sufficiently short geodesic and c is a curve with the same endpoints as γ, then* $\mathrm{Length}(\gamma) \leq \mathrm{Length}(c)$ *with equality if and only if $\gamma = c$.*

Proof. Suppose that γ is so short that it is fully contained inside a normal neighbourhood of its starting point $p := \gamma(0)$. We can also assume that c is contained inside this neighbourhood (otherwise c would be longer than γ because the length of γ is less than the radius of this neighbourhood while the length of the part of c between $c(0)$ and the point where c leaves this neighbourhood is larger.) Since $\exp_p$ is a diffeomorphism on this neighbourhood, we can uniquely write $c(s) := \exp_p(r(s)v(s))$ for some curve of unit vectors $v(s)$ and distance function $r(s)$. Note that $r(1) = \mathrm{Length}(\gamma)$ and $r(0) = 0$. See Figure **??**.

In the notation of the Gauss Lemma, $c(s) = f(s, r(s))$. Now we differentiate to find the tangent vector of c:

$$\frac{dc}{ds} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \circ (r(s))\frac{dr}{ds}.$$

Then we integrate to find the arc-length of c:

$$\mathrm{Length}(c) = \int_0^1 \|\dot{c}(s)\| ds$$

$$
= \int_0^1 \sqrt{\left\langle \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \circ (r(s))\dot{r}(s), \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \circ (r(s))\frac{dr}{ds} \right\rangle}\, ds
$$

$$
= \int_0^1 \sqrt{\left\| \frac{\partial f}{\partial s} \right\|^2 + |\dot{r}(s)|^2}\, ds
$$

because $\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle = 0$ by the Gauss Lemma and $\left\| \frac{\partial f}{\partial t} \right\| = 1$ by definition. But now,

$$
\text{Length}(c) = \int_0^1 \sqrt{\left\| \frac{\partial f}{\partial s} \right\|^2 + |\dot{r}(s)|^2} \geq \int_0^1 \dot{r}(s)\, ds = r(1) - r(0) = \text{Length}(\gamma)\, .
$$

If equality holds here, then it must be the case that $\frac{\partial f}{\partial s} = 0$ everywhere, in which case c is a monotone reparametrization of γ. □

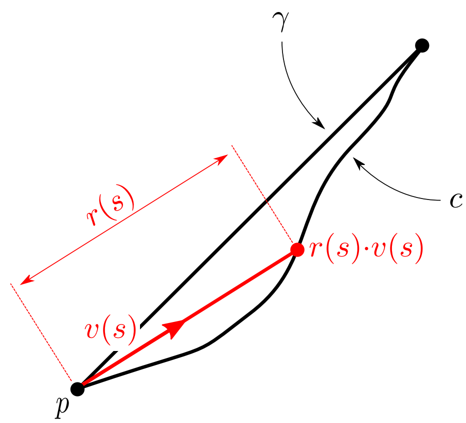

Figure 3: The picture of γ and c pulled back under $\exp_p$ into T_pS, and the quantities appearing in the proof of Theorem 1. Note that γ pulls back to a straight line!