CS 468

Differential Geometry for Computer Science

Lecture 9 — Intrinsic Geometry

Outline

From last lecture:

- The second fundamental form as extrinsic curvature.

Moving forward:

- The induced metric of a surface.
- Geodesics and length-minimizing curves.

Next time:

- The connection between the induced metric and geodesics.

Local Shape of a Surface

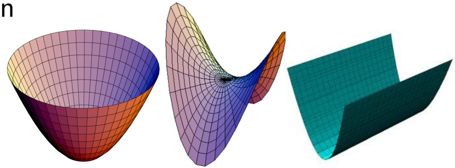
Example: Let S be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Without loss of generality, we can assume f vanishes to first order at $(0,0)$.

Then: The second fundamental form at $(0,0)$ is

$$A_{(0,0)} = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{\text{evaluated at } (0,0)}$$

And we can characterize the origin via the eigenvalues of $A_{(0,0)}$ as

- *Elliptic* — both > 0 or both < 0
- *Hyperbolic* — one of each sign
- *Parabolic* — one is zero,
- *Planar* — both are zero
- *Umbilic* — both are equal



Interpretation of the Mean Curvature

The mean curvature is the gradient of surface area.

- I.e. the area of the surface decreases fastest when it is deformed in the $H\vec{n}$ direction.

To see this:

- Let $\phi : \mathcal{U} \to \mathbb{R}^3$ parametrize S and let $f : \mathcal{U} \to \mathbb{R}$ be a function. Then $\phi_\varepsilon := \phi + \varepsilon f \cdot N$ parametrizes a deformation of S.
- Finally, let $g_\varepsilon(u) := [D\phi_\varepsilon]_u^\top [D\phi_\varepsilon]_u$ and $g := g_0$. Now:

$$\begin{aligned}\frac{d}{d\varepsilon}\text{Area}(\phi_\varepsilon(\mathcal{U}))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_{\mathcal{U}}\sqrt{\det(g_\varepsilon(u))}du\Big|_{\varepsilon=0}\\ &= \frac{1}{2}\int_{\mathcal{U}}\text{Tr}\left(g^{-1}\frac{dg_\varepsilon(u)}{d\varepsilon}\Big|_{\varepsilon=0}\right)\sqrt{\det(g(u))}\,du\\ &= -\int_{\mathcal{U}}H(u)f(u)\sqrt{\det(g(u))}\,du\end{aligned}$$

The Induced Metric

Observation: Let $\phi : \mathcal{U} \to \mathbb{R}^3$ parametrize a surface S. The object

$$g := [D\phi_u]^\top D\phi_u \qquad \text{for } u \in \mathcal{U}$$

has appeared quite often. What is the interpretation of g?

Definition: The object g is the induced metric of S.

- Let $E_i := \frac{\partial \phi}{\partial u^i}$ be the tangent vectors of S at $\phi(u)$.
- Then the components are $g_{ij} = E_i^\top E_j = \langle E_i, E_j \rangle$.
- Therefore the induced metric of a surface is the restriction of the Euclidean inner product to $T_{\phi(u)}S$, pulled back to $\mathcal{U}$ via ϕ.
- A parametrization gives you a representation of the intrinsic metric in the parameter plane as a matrix (actually a (2,0)-tensor).

Covariance

A scalar quantity defined on a surface S is “geometric” if its value computed w.r.t. any parametrization is always the same.

A different property holds for vector or tensor quantitities:

- The components of a “geometric” vector quantity computed w.r.t. two different parametrizations can be different.
- This is because the **basis** used to represent the quantity changes as well, and this must be taken into account.
- So we have **transformation formulas** for passing from one set of components to the other.
- This is called **covariance**.

Covariance of the Metric Tensor

Let $\phi : \mathcal{U} \to \mathbb{R}^3$ and $\psi : \mathcal{V} \to \mathbb{R}^3$ both parametrize S with $\phi(u) = \psi(v) = p \in S$. We get:

- $e_i := [0 \dots 1 \dots 0]^\top$ are the standard basis vectors in $\mathcal{U}$.
- $f_i := [0 \dots 1 \dots 0]^\top$ are the standard basis vectors in $\mathcal{V}$.
- $E_i := \frac{\partial \phi}{\partial u^i} = D\phi_u \cdot e_i$ and $F_i := \frac{\partial \psi}{\partial v^i} = D\psi_v \cdot f_i$ are bases for T_pS.

Then:

$$F_i = \frac{\partial \psi}{\partial v^i} = \frac{\partial \phi \circ \phi^{-1} \circ \psi}{\partial v^i} = \sum_j \frac{\partial [\phi^{-1} \circ \psi]^j}{\partial v^i} \frac{\partial \phi}{\partial u^j} = \sum_j \underbrace{\frac{\partial u^j}{\partial v^i}}_{\text{Change of basis matrix}} E_j$$

And

$$\langle F_k, F_\ell \rangle = \left\langle \sum_i \frac{\partial u^i}{\partial v^k} E_i, \sum_j \frac{\partial u^j}{\partial v^\ell} E_j \right\rangle = \sum_{ij} \frac{\partial u^i}{\partial v^k} \frac{\partial u^j}{\partial v^\ell} g_{ij}$$

The Geodesic Equation

Question: What is the shortest path between p, q in a surface S?

Fact: We can find an equation satisfied by the shortest path.

- Let $\gamma : I \to S$ be the shortest path and γ_ε a variation with variation vector field V that is tangent to S. Then

$$0 = \frac{d}{d\varepsilon}\text{Length}(\gamma_\varepsilon)\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_I \|\dot{\gamma}_\varepsilon(t)\|\, dt\bigg|_{\varepsilon=0} \quad \forall \text{ variations}$$

- From homework, we know that this implies

$$0 = \langle \vec{k}_\gamma, V\rangle \quad \forall \text{ variations} \qquad \Leftrightarrow \qquad \vec{k}_\gamma \perp S$$

Definition: Any curve satisfying this equation is called a geodesic.

The Geodesic Exponential Map

We'll see that the geodesic equation is a second-order ODE for γ. Thus there exists a unique local solution for every choice of

$$p := \gamma(0) \in S \quad \text{and} \quad X := \dot{\gamma}(0) \in T_p S$$

Definition: The assignment of (p, X) to a solution at distance one is called the geodesic exponential map and is denoted

$$\exp_p : B_\varepsilon(0) \subseteq T_p M \to M$$

$$\exp_p(X) := \begin{bmatrix} \text{one unit of arc-length} \\ \text{along the geodesic } \gamma \text{ with} \\ \gamma(0) = p \text{ and } \dot{\gamma}(0) = X \end{bmatrix}$$

Note: The geodesic itself is given by $\gamma(t) = \exp_p(tX)$.

Geodesics Locally Minimize Length

Two preliminary results...

Proposition: It is easy to see that $[D\exp_p]_0 = id$. Hence $\exp_p$ is a diffeomorphism near the origin in T_pM.

Proposition: ("Gauss lemma") Let $v, w \in T_v(T_pS)$. Then

$$\big\langle [D\exp_p]_v(v), [D\exp_p]_v(w) \big\rangle = \langle v, w \rangle$$

An important consequence...

Theorem: Geodesics locally minimize length: if γ is a sufficiently short geodesic and c is a curve with the same endpoints as γ, then

$$\text{Length}(\gamma) \leq \text{Length}(c)$$

with equality if and only if $\gamma = c$.

Hopf-Rinow Theorem

Some facts about geodesics:

- Length-minimizing curves are geodesics.
- Short geodesics are length-minimizing.
- There exist long geodesics that are not length-minimizing.

Next: We turn S into a metric space with distance function

$$d(p,q) := \inf_{\gamma \text{ from } p \text{ to } q} \text{Length}(\gamma)$$

Then d is continuous and satisfies the triangle inequality.

Hopf-Rinow Theorem: If exp is globally defined then any two points p, q can be connected by a geodesic with length $d(p,q)$.