CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 7 Student Notes: The Second Fundamental Form

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1 The unit normal vector of a surface.

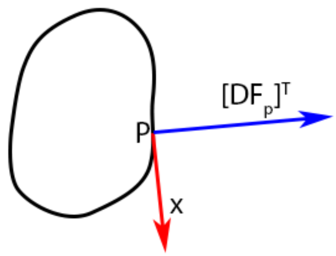

Figure 1: Normal vector of level set.

- The normal line is a geometric feature. The normal direction is not (think of a non-orientable surfaces such as a mobius strip). If possible, you want to pick normal directions that are consistent globally. For example, for a manifold, you can pick normal directions pointing out.
- Locally, you can find a normal direction using tangent vectors (though you can't extend this globally).
- Normal vector of a parametrized surface:

$$If \quad T_pS = span\{E_1, E_2\} \quad \text{then} \quad N := \frac{E_1 \times E_2}{\|E_1 \times E_2\|}$$

  This is just the cross product of two tangent vectors normalized by it's length. Because you take the cross product of two vectors, it is orthogonal to the tangent plane. You can see that your choice of tangent vectors and their order determines the direction of the normal vector.
- Normal vector of a level set:

$$N := \frac{[DF_p]^\top}{\|DF_p\|} \perp T_pS$$

  This is because the gradient at any point is perpendicular to the level set. This makes sense intuitively if you remember that the gradient is the "direction of the greatest increase" and that the value of the level set function stays constant along the surface. Of course, you also have to normalize it to be unit length.

2 Surface Area.

- We want to be able to take the integral of the surface. One approach to the problem may be to inegrate a parametrized surface in the parameter domain. This unfortunately doesn't work, and is dependent on the parametrization. (A square in the parameter domain won't necessarily be a square on the surface and squares of the same area in the parameter domain won't necessarily have the same area on the surface.)

- The correct solution is to use a Riemann sum that yields the surface area of a surface.
- Rieman sum that yields surface integral:

$$\int_S f\, d\mathsf{Area} := \lim \sum_i f(\phi(u_i)) \sqrt{\det(D\phi_{u_i}^\top D\phi_{u_i})}$$

  This is because the are of a rectangle is $E_1 \wedge E_2$ is $\|E_1 \times E_2\| = |\det(D\phi_u^\top D\phi_u)|^{1/2}$. You are taking the sum of infinitesimally small rectangles in coordinate space.

- Riemannian area form:

$$d\mathsf{Area}(u) := \sqrt{\det(D\phi_u^\top D\phi_u)}\, du^1\, du^2$$

  This is invariant of parametrization of the area integral. (We can prove this using the fact that changing variables in a integral results in a determinant.)

3 The Gauss map.

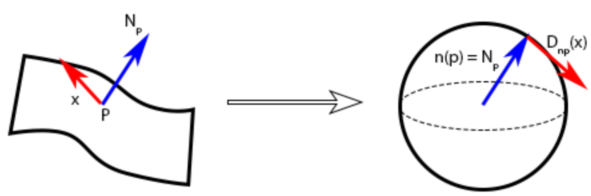

Figure 2: Map the normal vector onto a unit sphere.

- The Gauss map is the representation of each normal vector on a surface as a point on a unit sphere.
- Let S be an orientable surface with unit normal vector field n_p at each $p \in S$. The Gauss map of S is the mapping $N : S \to \mathbb{S}^2$ given by $N(p) := n_p$. Here we view the unit normal vector at a given $p \in S$ as a vector in $\mathbb{R}^3$ of length one and thus a point in $\mathbb{S}^2$.
- The Gauss map of a differentiable surface is itself differentiable. Thus we can study its differential $DN_p : T_pS \to T_{n_p}\mathbb{S}^2$.

  One should note that one needs to construct the normals in a consistent fashion for it to be differentiable (remember the normal line is a geometric feature, but the normal direction isn't). Also, since the surface must be differentiable, it doesn't apply to non-orientable surfaces like mobius strips.

- We can re-define $Dn_p : T_pS \to T_pS$ This is only legal because the normal vectors are the same in both representations and therefore the tangent planes are parallel.

4 Definition of the second fundamental form.

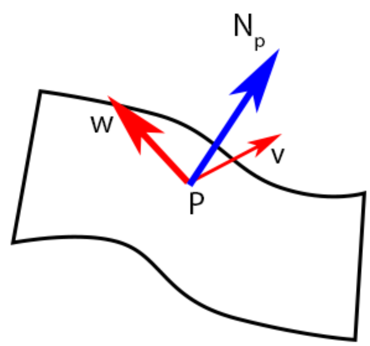

Figure 3: v, w are tangent vectors.

- Definition: the second fundamental form at $p \in S$ is the bilinear form $A_p : T_pS \times T_pS \to \mathbb{R}$ defined by $A_p(V, W) := -\langle DN_p(V), W\rangle$ for any $V, W \in T_pS$.
- $A_p(V, W)$ measure the projection onto W of the rate of change of N_p in the V-direction. By convention, there's a negative sign in the equation. (You'll see why in the next section.)
- A_p is self-adjoint, meaning swapping its inputs does not change the output.
- Proof that A_p is self-adjoint:

$$[A_p]_{ij} := \langle \frac{\partial N}{\partial u^i}, \frac{\partial \phi}{\partial u^j} \rangle$$
$$= -\frac{\partial}{\partial u^i}\langle N, \frac{\partial \phi}{\partial u^j}\rangle + \langle N, \frac{\partial^2 \phi}{\partial u^i \partial u^j}\rangle$$
$$= \langle N, \frac{\partial^2 \phi}{\partial u^i \partial u^j}\rangle$$
$$= \langle N, \frac{\partial^2 \phi}{\partial u^j \partial u^i}\rangle$$

- Note that changing the order of the tangent vectors doesn't matter, but that flipping the orientation of one of the vectors will change the sign.
- Note that if V,W are orthogonal $A_p(V, W)$ will be zero unless the surface has a "twist" at that point (meaning it locally looks like a saddle). See the next section for what happens when V,W are parallel.

5 The second fundamental form as extrinsic curvature.

- Let $c : [-1, 1] \to S$ be a curve in S with $c(0) = p$. Then the geodesic curvature vector of c at zero, $\vec{k}_c(0)$, is related to the second fundamental form at p as follows: $\langle \vec{k}_c(0), n_p\rangle = A_p(\dot{c}(0), \dot{c}(0))$. Note this is independent of $\ddot{c}$ or $c(t), \dot{c}(t)$ for $t \neq 0$.
- Proof:

$$\langle \vec{k}_c(0), N_p\rangle = \langle \ddot{c}(0), N_p\rangle$$
$$= \langle \frac{\partial}{\partial t}(\frac{\partial c}{\partial t}), N_p\rangle$$

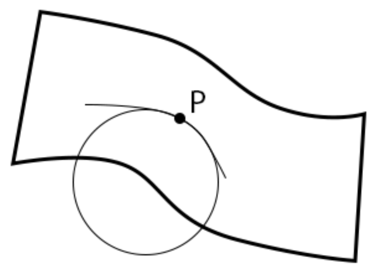

Figure 4: Locally fit a circle with the same curvature to the surface.

$$= \frac{\partial}{\partial t}\langle \frac{\partial c}{\partial t}, N_p\rangle - \langle \frac{\partial c}{\partial t}, \frac{\partial N_c(t)}{\partial t}\rangle$$
$$= -\langle \frac{\partial c}{\partial t}, \frac{\partial N_c(t)}{\partial t}\rangle$$
$$= -\langle \dot{c}(0), DN_p(\dot{c}(0))\rangle$$
$$= A_p(\dot{c}(0), \dot{c}(0))$$

- You can visualize this as fitting a circle with the same k_c to a point on the surface.
- Let V vary over all unit vectors in T_pS (the tangent vector space). Then $A_p(V, V)$ takes on a minimum value k_{min} and a maximum value k_{max}. These are the *principal curvatures* of S at p and are eigenvalues of A_p. The higher the curvature, the smaller the radius of the circle that matches the surface. Therefore, a curvature of 0, indicates that the surface is flat in that direction.
- The corresponding eigenvectors V_{min} and V_{max} are the *principal directions* of A_p. It's worth noting that (much like the normal direction) the principal direction of the surface at a point is ambiguous. If possible, you want to choose directions that are consistent throughout the surface. Doing so will give you the benefit of being able to tell which direction the surface is curved by the sign of the principle curvatures. Additionally, note that V_{min} and V_{max} are always orthogonal since they are eigenvectors. This means that N_p V_{min} and V_{max} are all orthogonal and can form a Darboux frame.
- There are several conventional ways of combining the principle curvatures.

  - Mean curvature: $H := k_{min} + k_{max}$ $(= \mathrm{Tr}(A_p)$ w.r.t. ONB). By convention, you don't divide by 2 even though it's called the "mean" curvature

  - Gauss curvature: $K := k_{min} \cdot k_{max}$ $(= \det(A_p)$ w.r.t. ONB).
- Mean curvature and Gauss curvature are just two methods of combining the principle curvatures. They have different characteristics. For example, if k_{min} is very small, then the Gauss curvature will be very small, but the mean curvature may not be depending on V_{max}. (This may not be entirely correct, but intuitively I like to think of the Gauss curvature as sort of like locally fitting a sphere to the surface and the mean curvature as fitting two orthogonal cylinders.)