CS 468

Differential Geometry for Computer Science

Lecture 5 — Surface Geometry

Outline

- The “official” definition of a surface.
- Examples.
- Tangent plane, normal vector.

Representing a Surface

Suppose you come across a surface in $\mathbb{R}^3$, what representation do you choose to describe it mathematically?

Each representation has its limitations.

- Not every surface is a graph.
- How do you find a level set function? Or if you know the level set function, how do you solve it? You have to solve equations! E.g. if $F(x, y, z) = 0$ you need to extract $z = g(x, y)$ with the property that $F(x, y, g(x, y)) = 0$.
- In general only part of a surface can be nicely parametrized.
- Non-uniqueness of all the representations.

The definition of a surface.

We would like a definition of a surface that as independent of representation as possible.

The method of choice: *local parametrizations*.

A a set $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$ there exists an open neighbourhood $V \subseteq \mathbb{R}^3$ containing p, an open neighbourhood $U \subseteq \mathbb{R}^2$ and a parametrization $\sigma : U \to V \cap S$ such that:

1. $\sigma = (\sigma^1, \sigma^2, \sigma^3)$.
2. σ is invertible as a map from U onto $V \cap S$ and has a continuous inverse.
3. $D\sigma_q$ is injective $\forall\ q$. (If and only if $\det\big((D\sigma_q)^\top D\sigma_q\big) \neq 0$.)

Examples

- A graph is a regular surface.
- Proof that the sphere is a regular surface by writing it as the union of six graphs over the coordinate planes.
- Another example where the coordinates are differentiable at q but $D\sigma_q$ is non-injective: the sphere in polar coordinates.

A more sophisticated example.

- The inverse image of a *regular value* is regular surface...

Regular Values

Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. A value $c \in \mathbb{R}$ is called *regular* if $DF_p \neq 0$ for all $p \in F^{-1}(c)$.

E.g. a non-regular value: $F(x, y, z) := x^2 + y^2 \pm z^2$ and $c = 0$.

Theorem: $F^{-1}(c)$ is a regular surface.

Proof:

- Here $F(p) = 0$ and $DF_p \neq 0$ meaning $\exists\, i$ so that $\frac{\partial F(p)}{\partial x^i} \neq 0$.
- W.l.o.g. $i = n$ so we get from the Im. F. T. the local solution $x^n = g(\bar{x})$ where $\bar{x} := (x^1, \ldots, x^{n-1},)$ so that $F(\bar{x}, g(\bar{x})) = 0$.
- Now $F^{-1}(0)$ near p projects down onto an open set U in the $\bar{x}$-plane and is equal to the graph $\{(\bar{x}, g(\bar{x})) : \bar{x} \in U\}$.
- Thus it's a surface!

The Tangent Space of a Surface

- Curves in a surface. The coordinate curves.
- Tangent vectors to a surface.

The "official" definition.

> Let $\sigma : U \subseteq \mathbb{R}^2 \to V \cap S \subseteq \mathbb{R}^3$ be a parametrization of a subset of a surface S and let $p \in S$ such that $p = \sigma(u)$ for some $u \in U$.
>
> The tangent plane T_pS defined as $Image(D\sigma_u) \subseteq T_{\sigma(u)}\mathbb{R}^3$.

A general principle of differential geometry is at work here:

- We define a *geometric concept* using a parametrization... then we must show independence of the chosen parametrization.

Parameter Independence of the Tangent Space

- The previous definition depends on the parametrization σ.
- What if we change parametrization?
- We get the same tangent space!!

The proof would go like this:

→ Let $\sigma : U \to \mathbb{R}^3$ and $\tau : U' \to \mathbb{R}^3$ be two different parametrizations of the same part of the surface.

→ Now $\sigma \circ \tau^{-1} : U' \to U$ is a smooth bijection.

→ Then we compute

$$\begin{aligned} Image\big(D\sigma_u\big) &= Image\big(D(\sigma \circ \tau^{-1} \circ \tau)_u\big) \\ &= Image\big(D(\sigma \circ \tau^{-1})_{\tau(u)} \cdot D\tau_u\big) \\ &= Image\big(D\tau_u\big) \end{aligned}$$

Basis for the Tangent Space

- This is NOT a geometric concept.
- Three ways of getting a basis for the tangent space:
  - → Tangent space of a parametric surface.
  - → Tangent space of a graph.
  - → Tangent space of a level set.

    The relation $F \circ c(t) = const.$ for curves $c(t)$ belonging to the level set $F^{-1}(const.)$
- Note where the construction breaks down!