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CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 4 Note: Surface Theory I

Before we go into the definition of surfaces, we will review some important background knowledge including the differential of a function and the Inverse and Implicit Function Theorems.

The Differential of a Function

(a) **Tangent Spaces**

We will restrict ourselves in Euclidean spaces. Consider the space $\mathbb{R}^n$. Denote $T_p\mathbb{R}^n$ the *tangent spaces of* $\mathbb{R}^n$ *at* p. What it menas here is that for *each* and *every* point p in $\mathbb{R}^n$, we introduce a new coordinate system where all the vectors originated at p will reside in. These coordinate frames can either be moving around as we saw with the Frenet and Bishop frames, or they can be fixed as the standard basis.

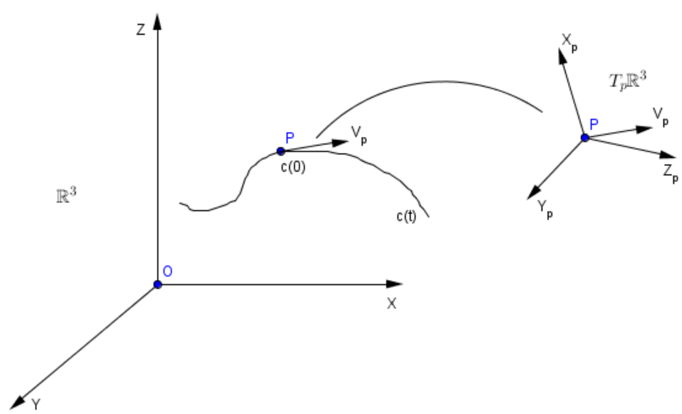

Figure 1: The space $\mathbb{R}^3$ and a tangent space at point $p \in \mathbb{R}^3$.

The important part here is that given any vector in the tangent space at p, we can always find a curve passing through p and having the vector as its tangent at p. More formally, given $V_p \in T_p\mathbb{R}^n$, we can find a curve $c : I \to \mathbb{R}^n$ with $c(0) = p$ and $c'(0) := \left.\dfrac{dc(t)}{dt}\right|_{t=0} = V_p$. For example, we can find $c(t) = tV_p + p$, a straight line with slope V_p. Of course this is not unique.

(b) **The Differential of a Function**

Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f(x^1, \ldots, x^n) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$.

The differential of f at p, Df_p, is the $m \times n$ matrix defined by

$$Df_p := \begin{pmatrix} \dfrac{\partial f^1}{\partial x^1} & \dfrac{\partial f^1}{\partial x^2} & \cdots & \dfrac{\partial f^1}{\partial x^n} \\ \dfrac{\partial f^2}{\partial x^1} & \dfrac{\partial f^2}{\partial x^2} & \cdots & \dfrac{\partial f^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f^m}{\partial x^1} & \dfrac{\partial f^m}{\partial x^2} & \cdots & \dfrac{\partial f^m}{\partial x^n} \end{pmatrix}$$

As we all know, a matrix can be thought of as a linear transformation. Here as well, we can think of Df_p as a linear mapping $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$. In other words, Df_p maps a vector in the tangent space at the source point p to a vector in the tangent space at the target point $f(p)$. More formally,

$$\frac{d}{dt} f(c(t))\bigg|_{t=0} = Df_p V_p$$

Why is this true? Let's suppose we consider the curve $c : [-\epsilon, \epsilon] \to \mathbb{R}^n$ locally near p and let $\dfrac{dc(t)}{dt}\bigg|_{t=0} = V_p = (a_1, a_2, \ldots, a_n)^T$ in the standard basis. Apply forward f mapping. The whole curve $c(t)$ is mapped to $f(c(t))$ and in particular the point p is mapped to $f(p)$. The tangent at $t = 0$ in the domain must also be mapped to the tangent at $t = 0$ in the image as well. This means that the tangent at p, i.e. V_p, is mapped to $\dfrac{d}{dt} f(c(t))\bigg|_{t=0} \in T_{f(p)}\mathbb{R}^m$, which is the tangent at $f(c(0)) = f(p)$ in the image. Now, by applying the chain rule, we can calculate (in the standard basis)

$$\begin{aligned} \frac{d}{dt} f(c(t))\bigg|_{t=0} &= \begin{pmatrix} \dfrac{d}{dt} f^1(c(t))\bigg|_{t=0} \\ \vdots \\ \dfrac{d}{dt} f^m(c(t))\bigg|_{t=0} \end{pmatrix} \\ &= \begin{pmatrix} \displaystyle\sum_{i=1}^n \frac{df^1(c(t))}{dx^i} \frac{dc^i(t)}{dt}\bigg|_{t=0} \\ \vdots \\ \displaystyle\sum_{i=1}^n \frac{df^m(c(t))}{dx^i} \frac{dc^i(t)}{dt}\bigg|_{t=0} \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{i=1}^n \frac{df^1(p)}{dx^i} a_i \\ \vdots \\ \displaystyle\sum_{i=1}^n \frac{df^m(p)}{dx^i} a_i \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{\partial f^1(p)}{\partial x^1} & \cdots & \dfrac{\partial f^1(p)}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f^m(p)}{\partial x^1} & \cdots & \dfrac{\partial f^m(p)}{\partial x^n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = Df_p V_p \end{aligned}$$

This justifies our claim that Df_p is the linear map between the two tangent spaces.

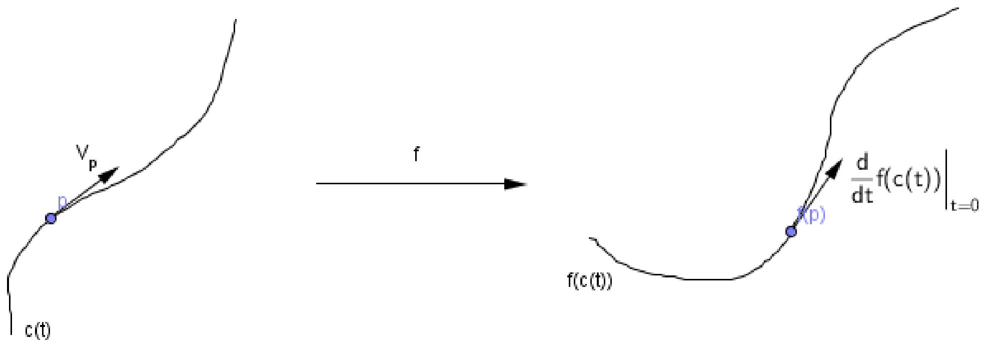

Figure 2: The forward mapping of f

(c) **The Rank of the Differential**

The differential of a function f can say a lot about local properties of the map f. In particular, if f has *locally constant rank* K on an open $\Omega \subseteq \mathbb{R}^n$, i.e. $\mathbf{rank}(Df_p) = K, \forall p \in \Omega$, then the behavior of Df_p carries over to f in just the right way. Notice that this is true only locally. That is if Df_p has constant rank in a small neighborhood, then Df_p dictates what f should look like near p. We call this the *Rank Theorems*, which is stated below

Theorem (Rank Theorems) We can locally "modify" f, i.e. there exists a change of coordinates on the domain and the range of f, into an equivalent map $\tilde{f}$ such that

Case 1 Df_p is injective for all $p \in \Omega \subseteq \mathbb{R}^n$, then $n \leq m$ and

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0)$$

Case 2 Df_p is surjective for all $p \in \Omega \subseteq \mathbb{R}^n$, then $n \geq m$ and

$$\tilde{f}(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) = (x^1, \ldots, x^m)$$

Case 3 Df_p is bijective for all $p \in \Omega \subseteq \mathbb{R}^n$, then $n = m$ and

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^m)$$

Case 4 Df_p has rank k for all $p \in \Omega \subseteq \mathbb{R}^n$ where $k < \min(n, m)$, then

$$\tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0)$$

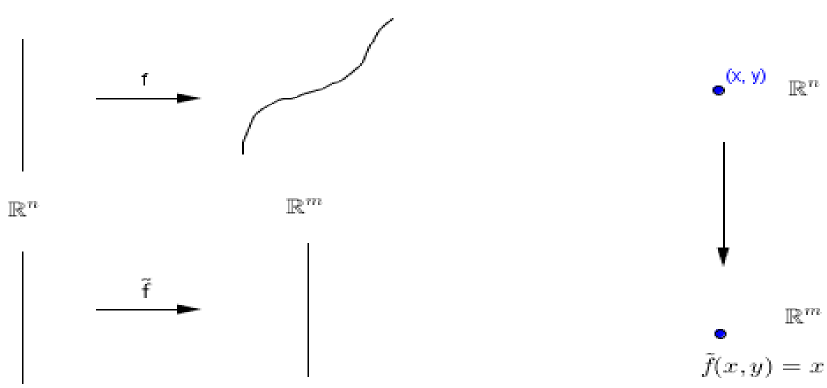

Figure 3: (Left) Straightening in Case 1 (Right) Projection in Case 2

In case 1, we can picture $\tilde{f}$ as straightening out the line. Consider an example of a curve, a map from $\mathbb{R}$ to $\mathbb{R}^3$. By changing a coordinate system, we can make a curvy line into a straight line. In case 2, we can picture it as a projection from $\mathbb{R}^n$ to $\mathbb{R}^m$. Case 3 is the combination of case 1 and 2, for a bijection is both injective and surjective. And finally in case 4, by saying k is strictly less than n and m, we are saying that Df_p is neither injective nor surjective.

We will not go into the proof of the Rank Theorems. Instead, in the next section, we will state two important theorems in multivariable analysis which the proof relies on.

The Inverse and Implicit Funcion Theorems

Theorem (Inverse Function Theorem) If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with Df_p bijective, then f is invertible on a neighborhood of p.

Note that Df_p is bijective at p if and only if $\det((Df_p)^T Df_p) \neq 0$. Our takeaway from the previous section still applies to this theorem, i.e. f behaves similarly to Df_p locally.

Theorem (Implicit Function Theorem) If $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $D_2F_{(p,q)}$ bijective and $F(p,q) = 0$, then the equation $F(x,y) = 0$ can be solved for points (x,y) near (p,q) in the following sense

(i) There exists a function $g : \mathbb{R}^k \to \mathbb{R}^n$ defined near q such that $q = g(p)$ and also $F(x, g(x)) = 0$.

(ii) We can compute Dg_x in terms of $D_1F_{(x,g(x))}$ and $D_2F_{(x,g(x))}$.

Here D_1 and D_2 are defined similarly to the differential D but are differentiated with respect to the coordinates of x and y, respectively, instead. To understand this theorem, let's consider a simple example.

<u>Example</u> Consider $F(x,y,z) = x^2 + y^2 + z^2 - 1$. Here $k + n = 3, n = 1$. The solution of $F(x,y,z) = 0$ is the unit sphere in $\mathbb{R}^3$. Suppose we want to eliminate z. We can find $z = \pm\sqrt{1 - x^2 - y^2}$. So locally we can define $g(x,y) = \sqrt{1 - x^2 - y^2}$ and the solution becomes $(x, y, \sqrt{1 - x^2 - y^2})$. How does this related to the Implicit Function Theorem? We can see that $D_2F = \dfrac{\partial F}{\partial z} = 2z$ is invertible when $z \neq 0$. So we can solve the equation $F(x,y,z) = 0$ locally provided $z \neq 0$, i.e. $z = \pm\sqrt{1 - x^2 - y^2}$ and find $g(x,y) = \sqrt{1 - x^2 - y^2}$ where $F(x, y, g(x,y)) = 0$ for (x,y) near (p,q). This makes sense since when $z = 0$, the tangent becomes vertical and the derivative is non-defined.

Three Kinds of Surfaces

For simplicity, we will focus our interest in $\mathbb{R}^3$. The common representations of surfaces in $\mathbb{R}^3$ are as follows

(1) *Graphs of Functions* We can represent a surface as a graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. That is for every $(x,y) \in \mathbb{R}^2$, the point $(x, y, f(x,y))$ lies on the surface in $\mathbb{R}^3$. This type of representation has its limitation, however, because not every surface is a graph of a function. One simple example is a sphere since it has both the upper hemisphere and the lower hemisphere, so each point on the plane will be mapped to two different points.

(2) *Level Sets of Functions* Alternatively, we can represent a surface using a function $F : \mathbb{R}^3 \to \mathbb{R}$. That is the surface will be a solution to the equation $F(x,y,z) = 0$. As in the example in the previous section, the solution to $F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ is the unit sphere in $\mathbb{R}^3$. Note that by the Implicit Function Theorem, we can find a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that locally $F(x, y, g(x,y)) = 0$ and the points on the surface are $(x, y, g(x,y))$.

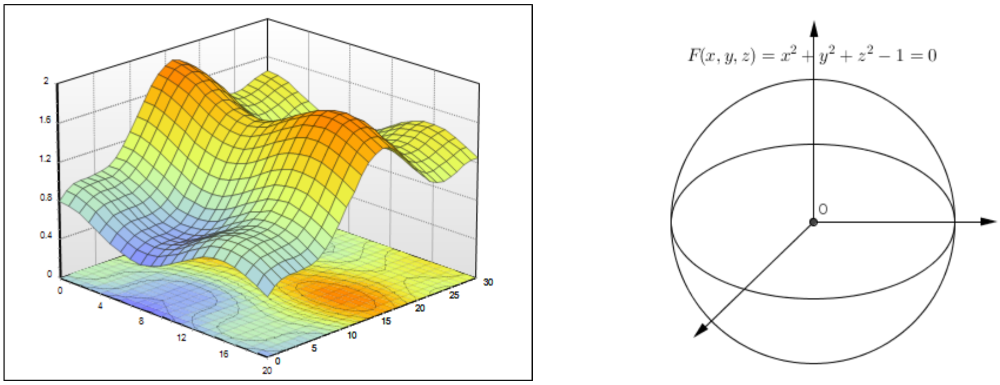

Figure 4: (Left) Surface as a graph of a function. (Right) A sphere cannot be represented as a graph but can be represented as a level set of a function.

However, this method still has limitations since not all equations can be solved analytically nor given an arbitrary surface can we find such a function F where the surface will be its level set.

(3) *Parametric Surfaces* This is a generalization of parametric curves. A surface $\sigma : U \to \mathbb{R}^3$ where $U \subseteq \mathbb{R}^2$ is an open domain in the plane and

$$\sigma(u^1, u^2) := (\sigma^1(u^1, u^2), \sigma^2(u^1, u^2), \sigma^3(u^1, u^2))$$

Again, we use the example of a unit sphere. By introducing the azimuthal angle θ and the polar angle φ, we can parametrized the unit sphere by

$$\sigma(\theta, \varphi) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$$

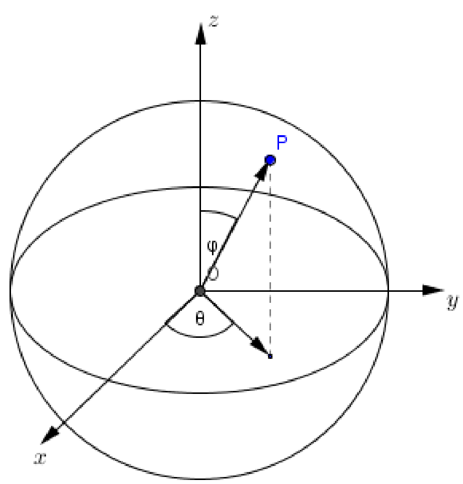

Figure 5: Spherical parametrization of a unit sphere