http://www.alvinomassage.com/images/knot.jpg

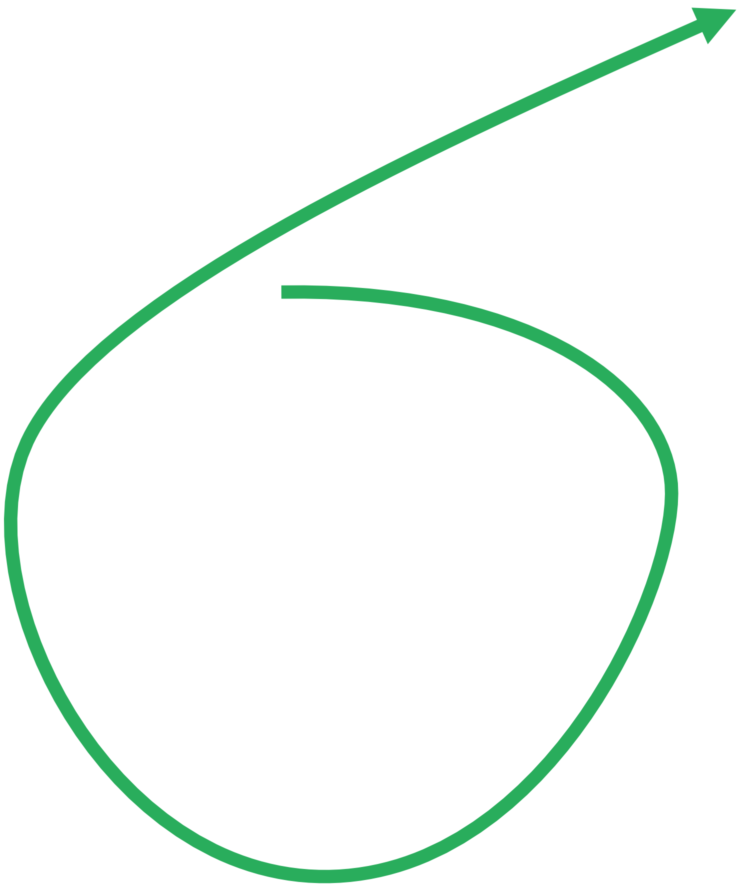
Discrete Curves

CS 468, Spring 2013
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher

<review>

Review



$$\gamma(t)$$

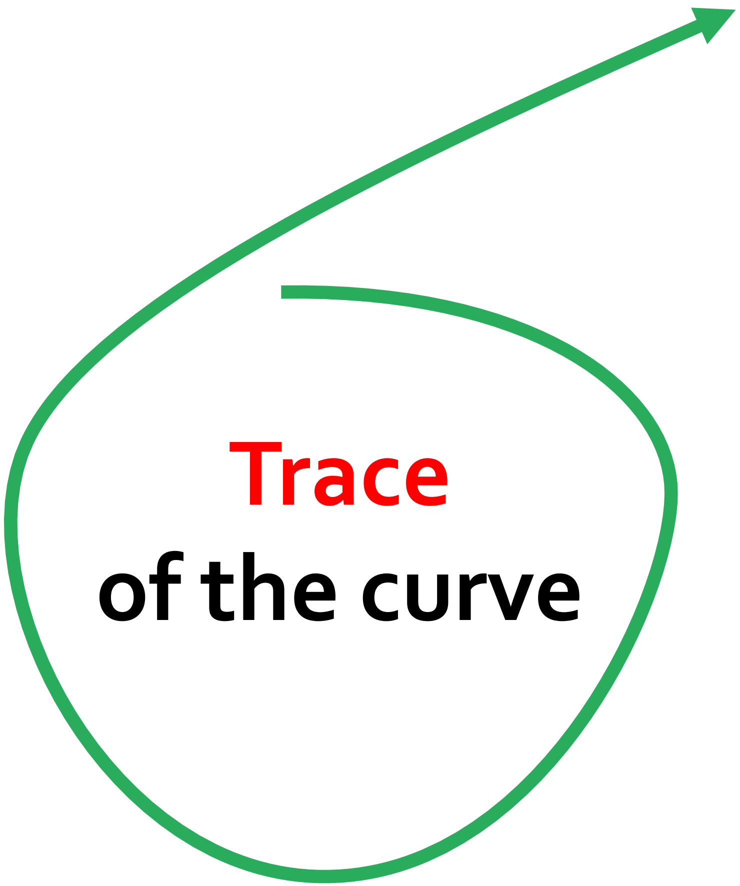
Function of time

$$s(t) = \int_0^t \|\gamma'\| \, dt$$

$$\gamma(s)$$

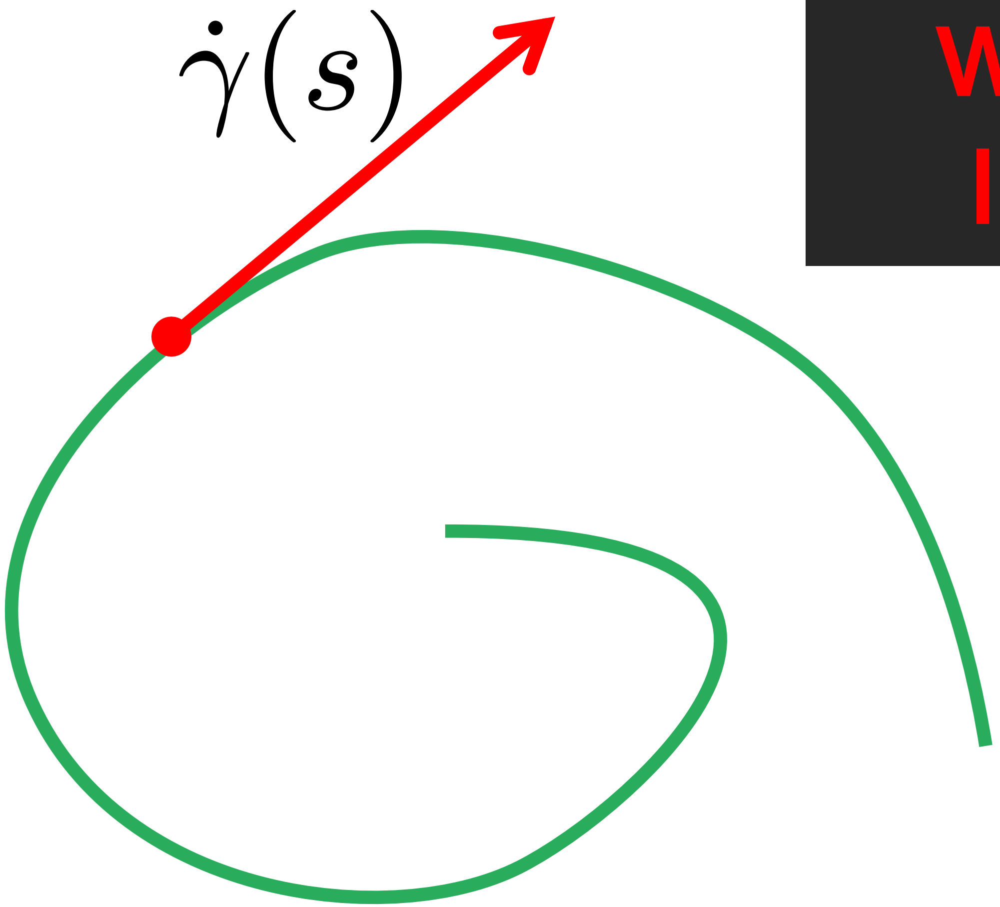
Function of distance

The *Geometric* Object



$$\{\gamma(t) : t \in \mathbb{R}\}$$

Unit Tangent

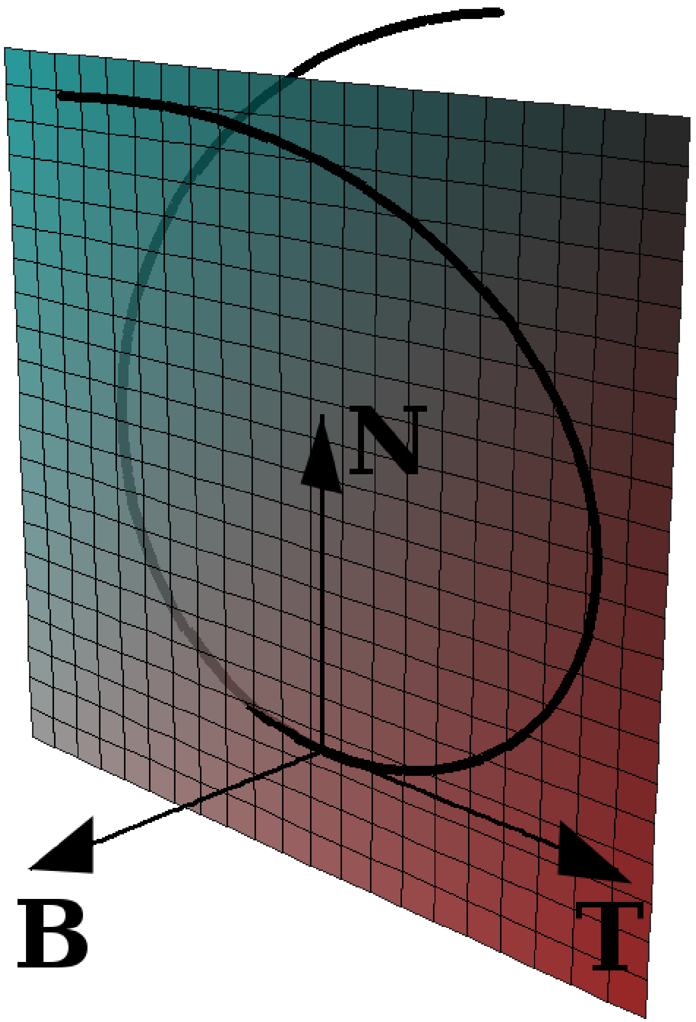


Intuition: Why unit length?

Quick Exercise

Take $v(t) : \mathbb{R} \to \mathbb{R}^n$ such that $\|v(t)\| = 1 \ \forall t$. Show $\langle v(t), v'(t) \rangle = 0 \ \forall t$.

Frenet Frame



$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

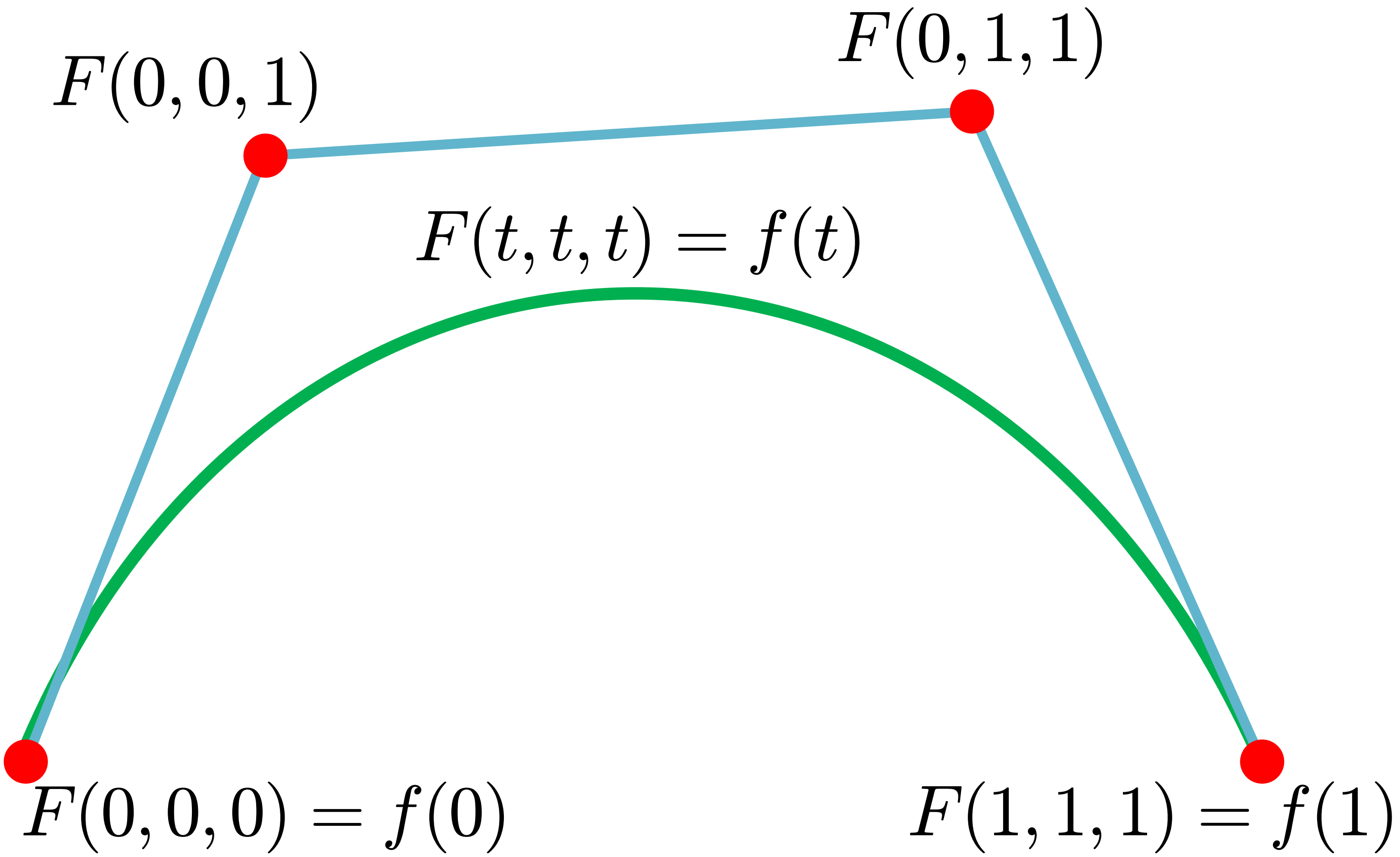
$$\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

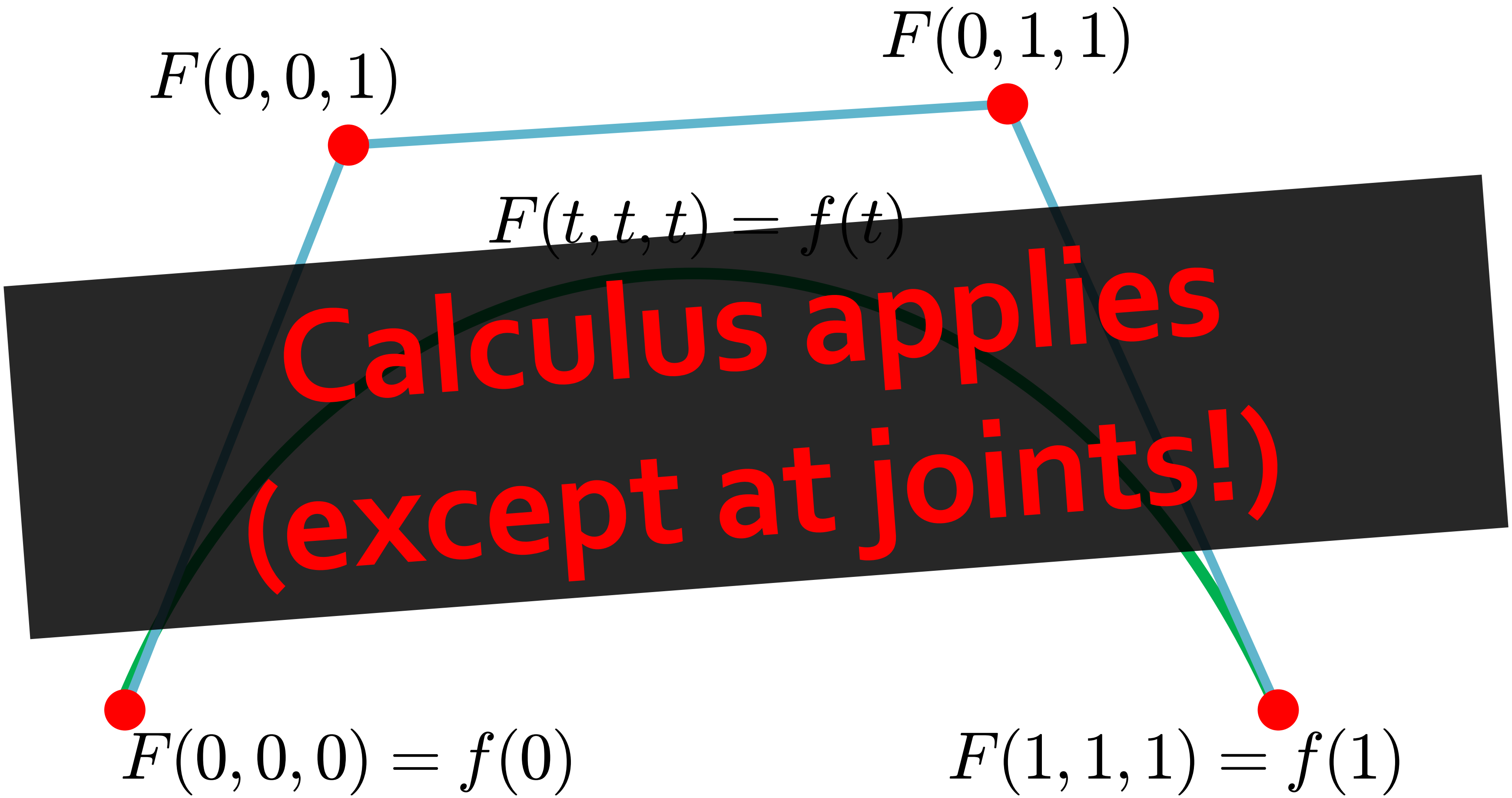
</review>

Old-School Approach



Piecewise smooth approximations

Old-School Approach



Piecewise smooth approximations

Question

What is the arc length of a cubic Bézier curve?

$$s = \int_{t_0}^{t_1} \sqrt{\gamma_x'^2 + \gamma_y'^2 + \gamma_z'^2}\; dt$$

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What is the arc length of a cubic Bézier curve?

$$s = \int_{t_0}^{t_1} \sqrt{\gamma_x'^2 + \gamma_y'^2 + \gamma_z'^2}\, dt$$

Hint: It's usually impossible.

Sad fact:

Closed-form expressions rarely exist. When they do exist, they usually are messy.

Only Approximations Anyway

$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \to \mathbb{R}^3\}$$

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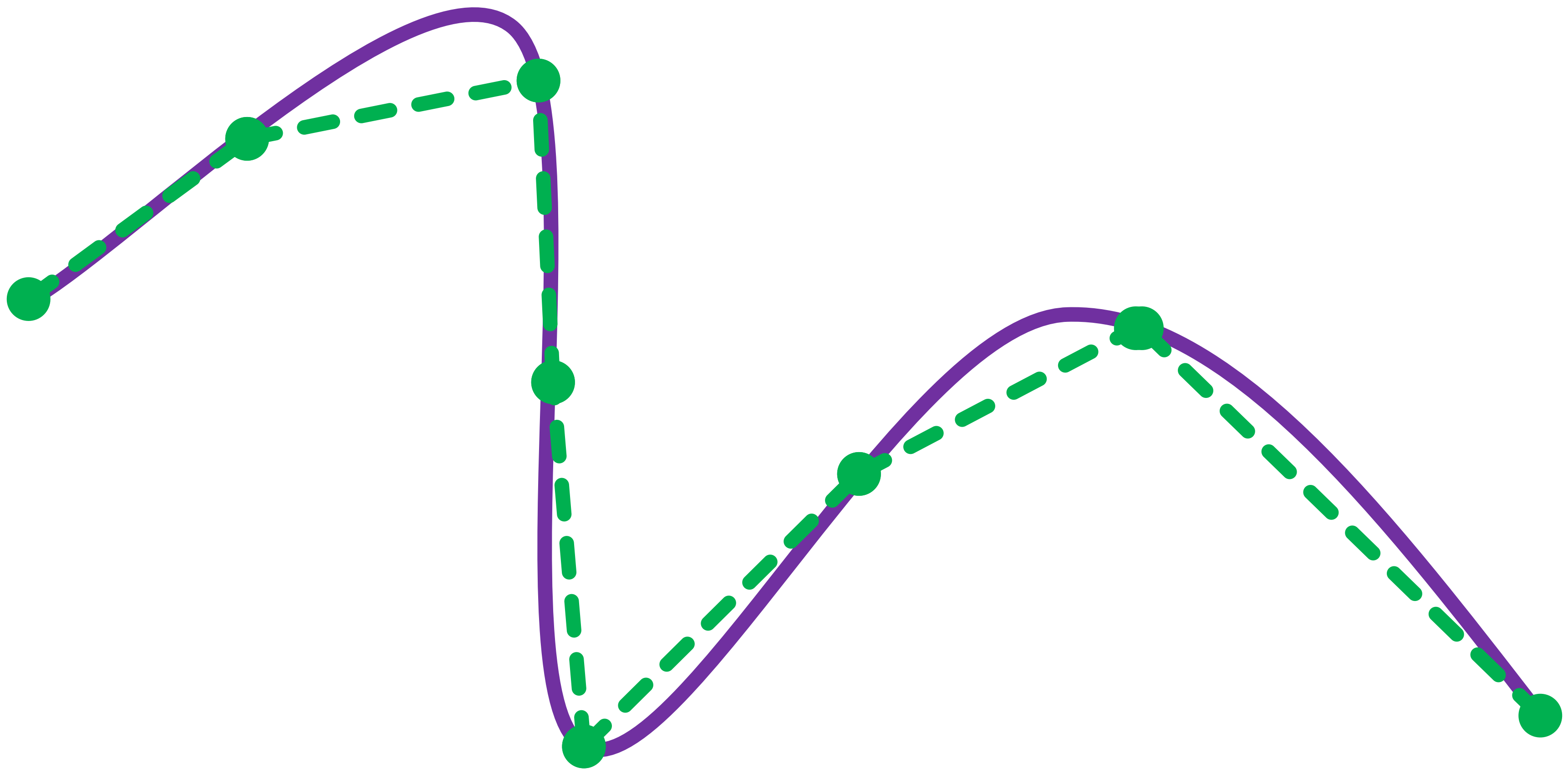
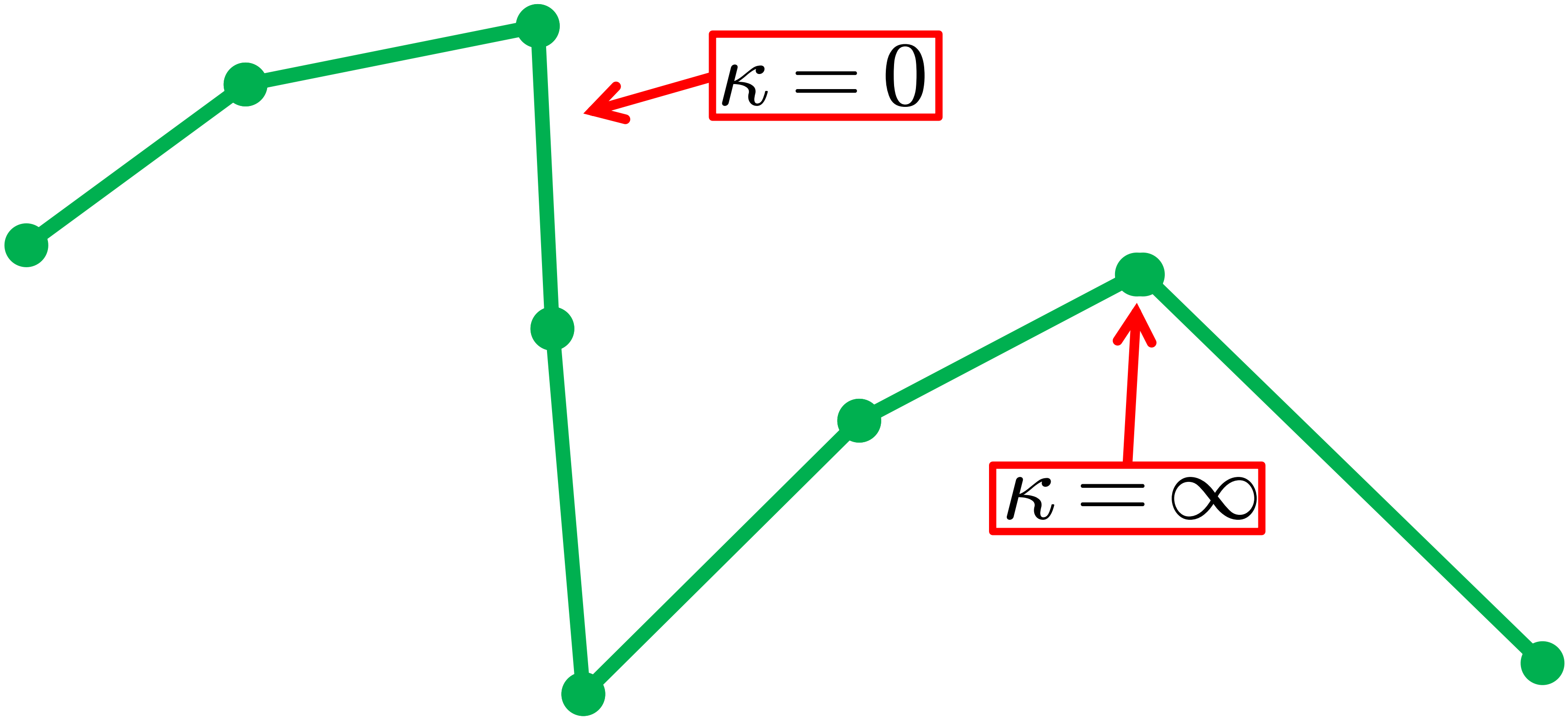
$$\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \to \mathbb{R}^3\}$$

Equally Good Approximation



Piecewise linear

Big Problem



Boring differential structure

Finite Difference Approach

$$f'(x) \approx \frac{1}{h}(f(x+h) - f(x))$$

THEOREM: As $\Delta h \rightarrow 0$, [insert statement].

Finite Difference Approach

f … $)$

$$h \neq 0$$

THEO… …ment].

Two Key Considerations

- Convergence to continuous theory
- Discrete behavior

Today's Goal

Examine discrete theories of differentiable curves.

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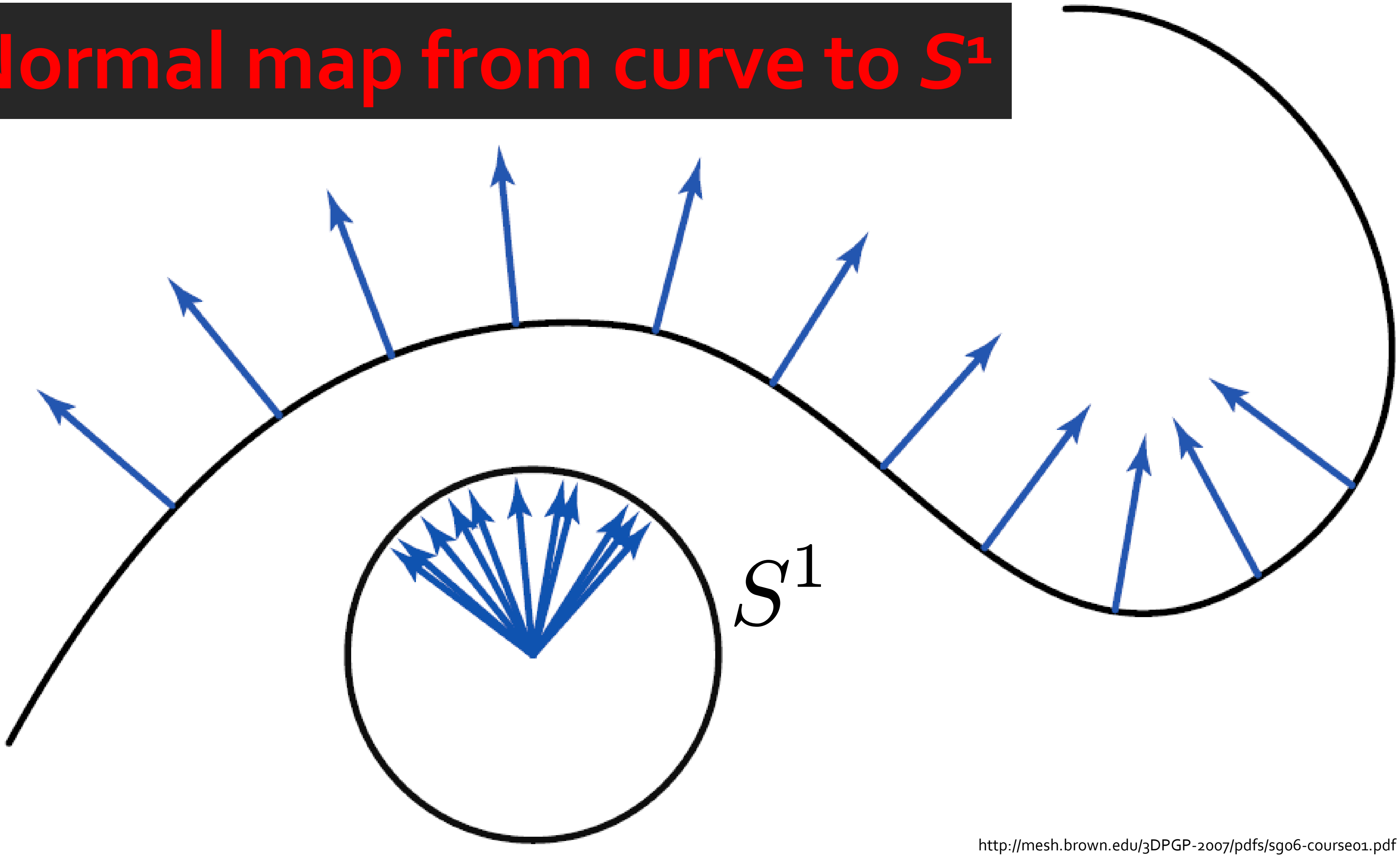
Examine discrete theor<u>ies</u> of differentiable curves.

Gauss Map

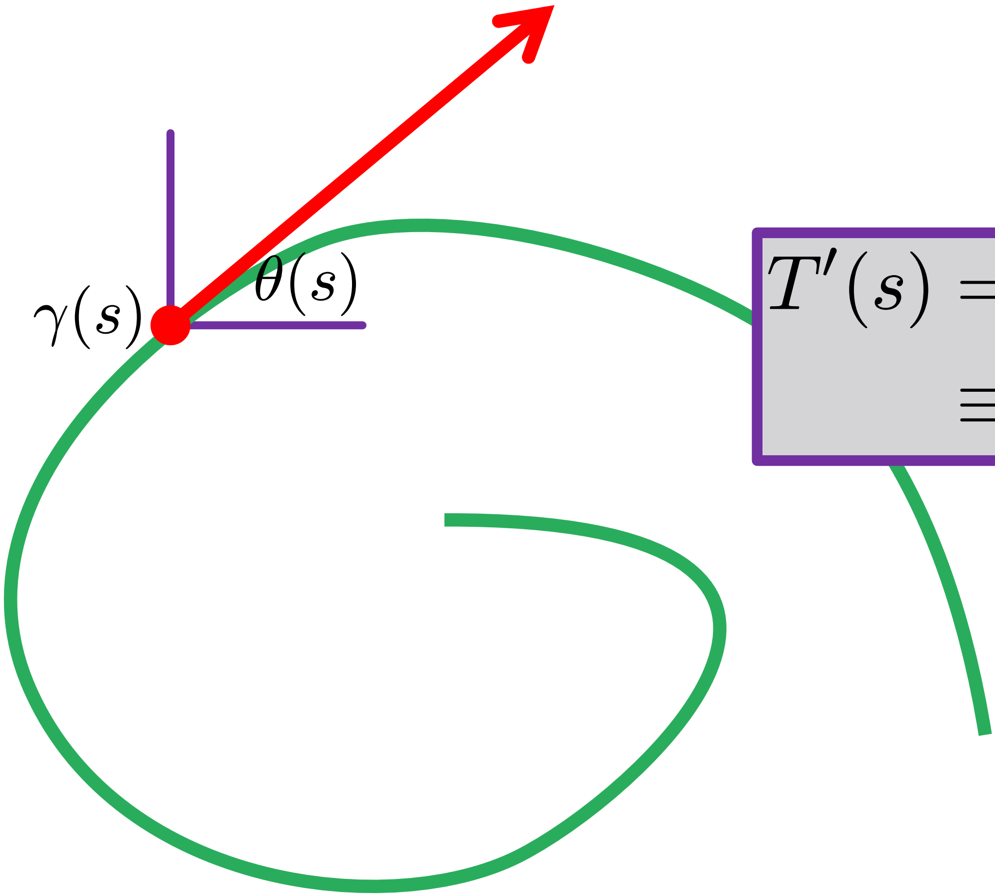
Normal map from curve to S^1



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

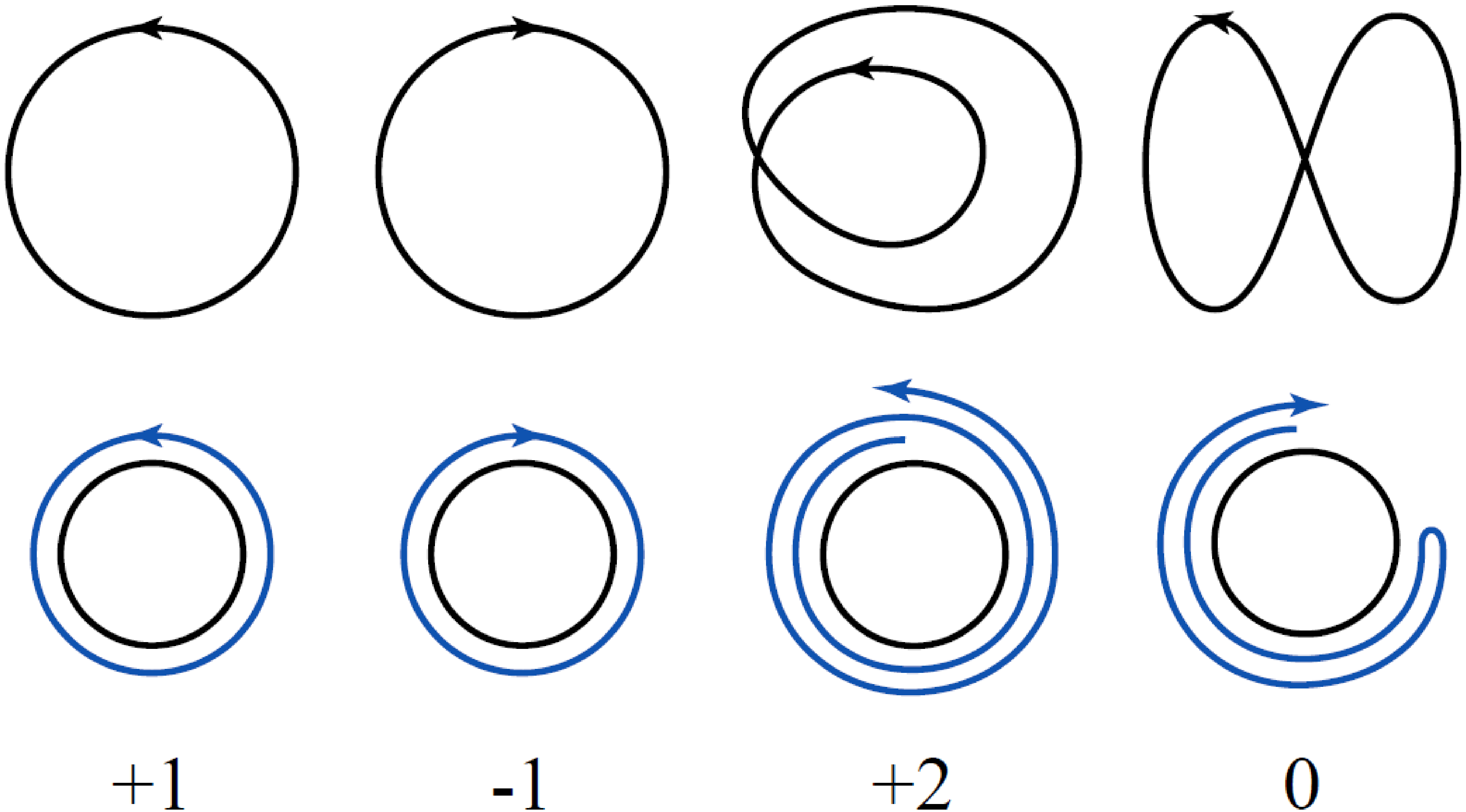
Signed Curvature on Plane Curves

$$T(s) = (\cos\theta(s), \sin\theta(s))$$



$$T'(s) = \theta'(s)(-\sin\theta(s), \cos\theta(s))$$
$$\equiv \kappa(s)N(s)$$

Turning Numbers



http://mesh.brown.edu/3DPGP-2007/pdfs/sg06-course01.pdf

Recovering Theta

$$\theta'(s) \equiv \kappa(s)$$

$$\Downarrow$$

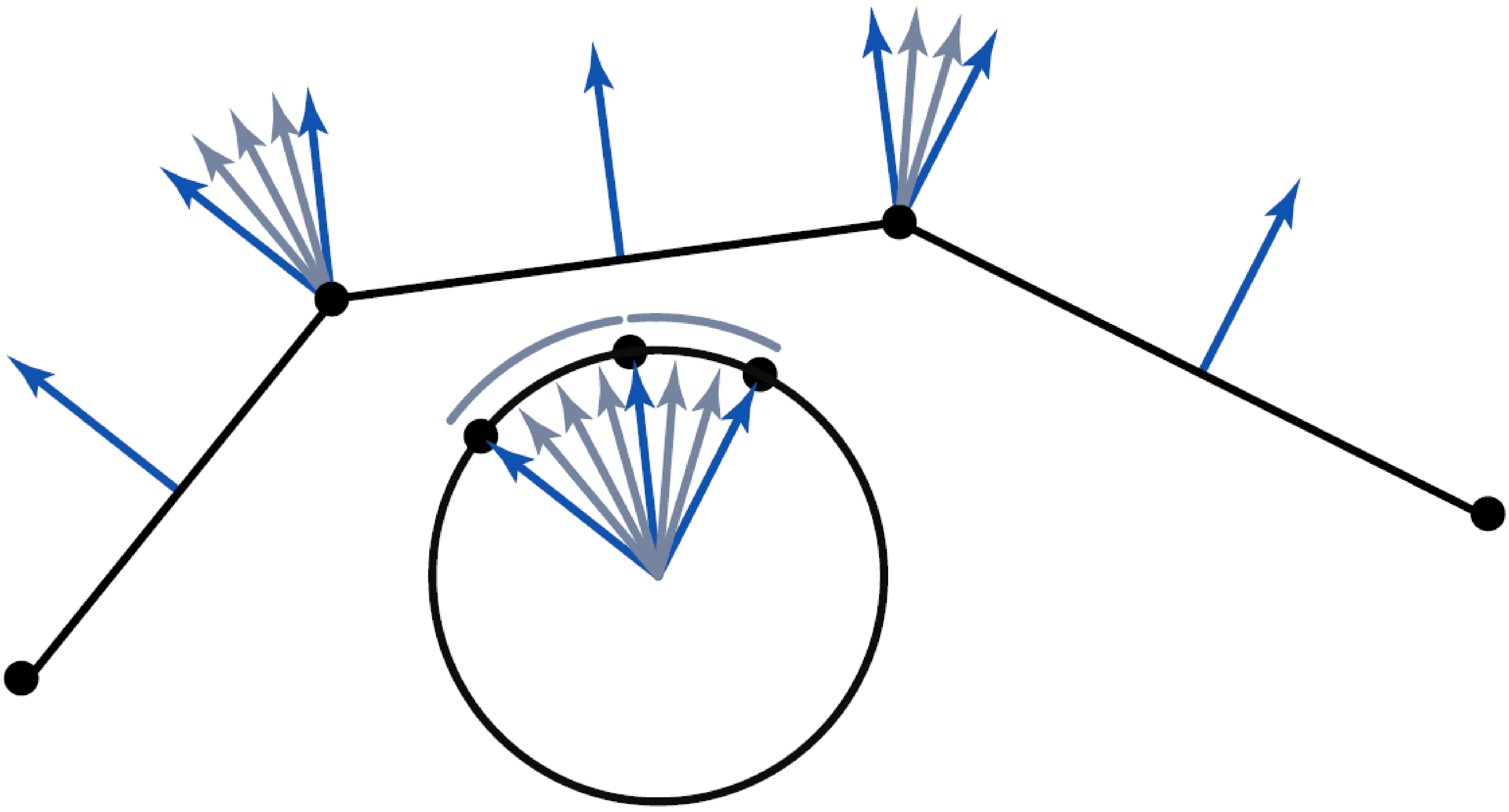
$$\Delta\theta = \int_{s_0}^{s_1} \kappa(s)\ ds$$

Turning Number Theorem

$$\boxed{\int \kappa(s)\ ds = 2\pi k}$$

A "global" theorem!

Discrete Gauss Map



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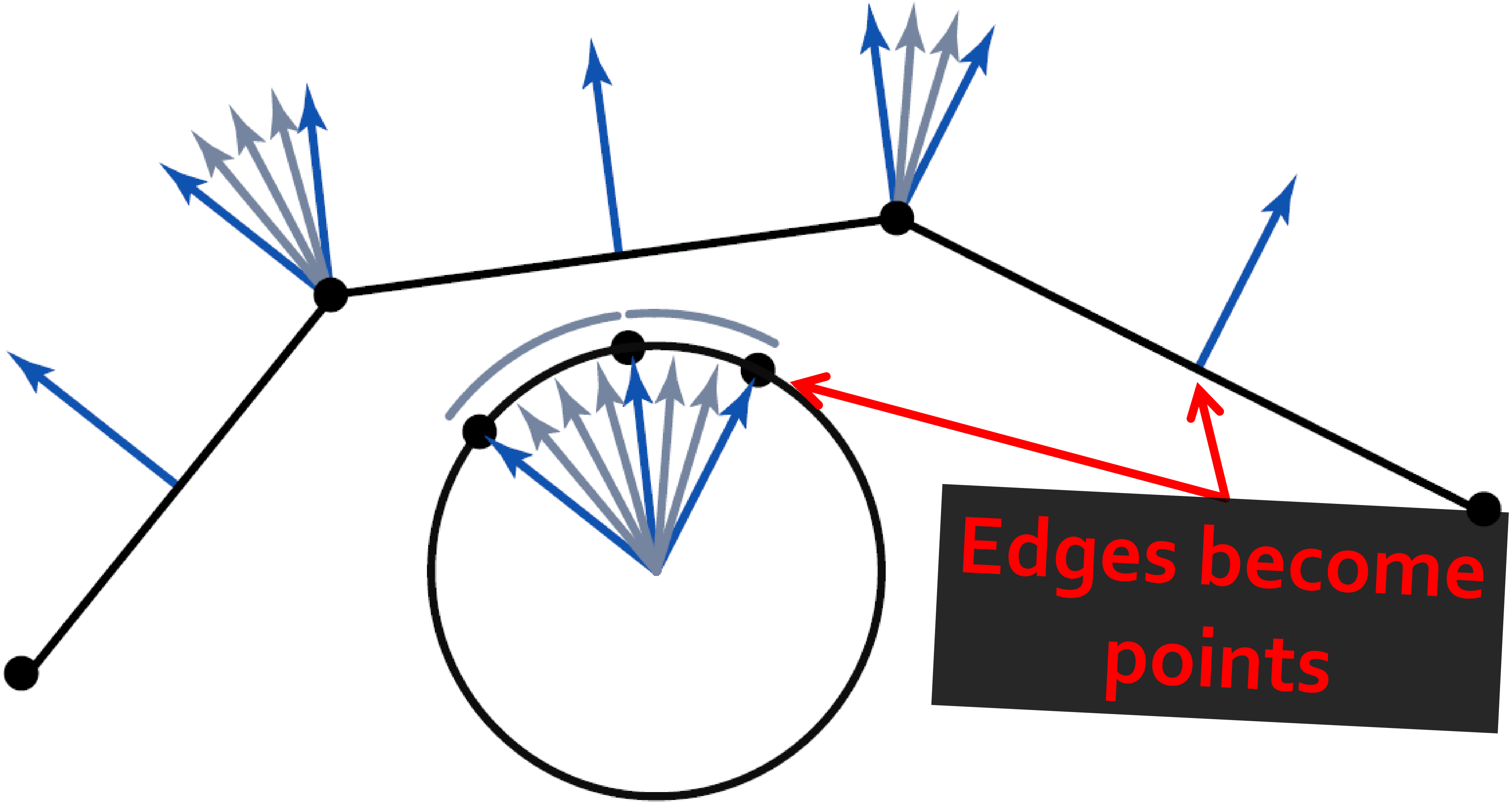
Discrete Gauss Map



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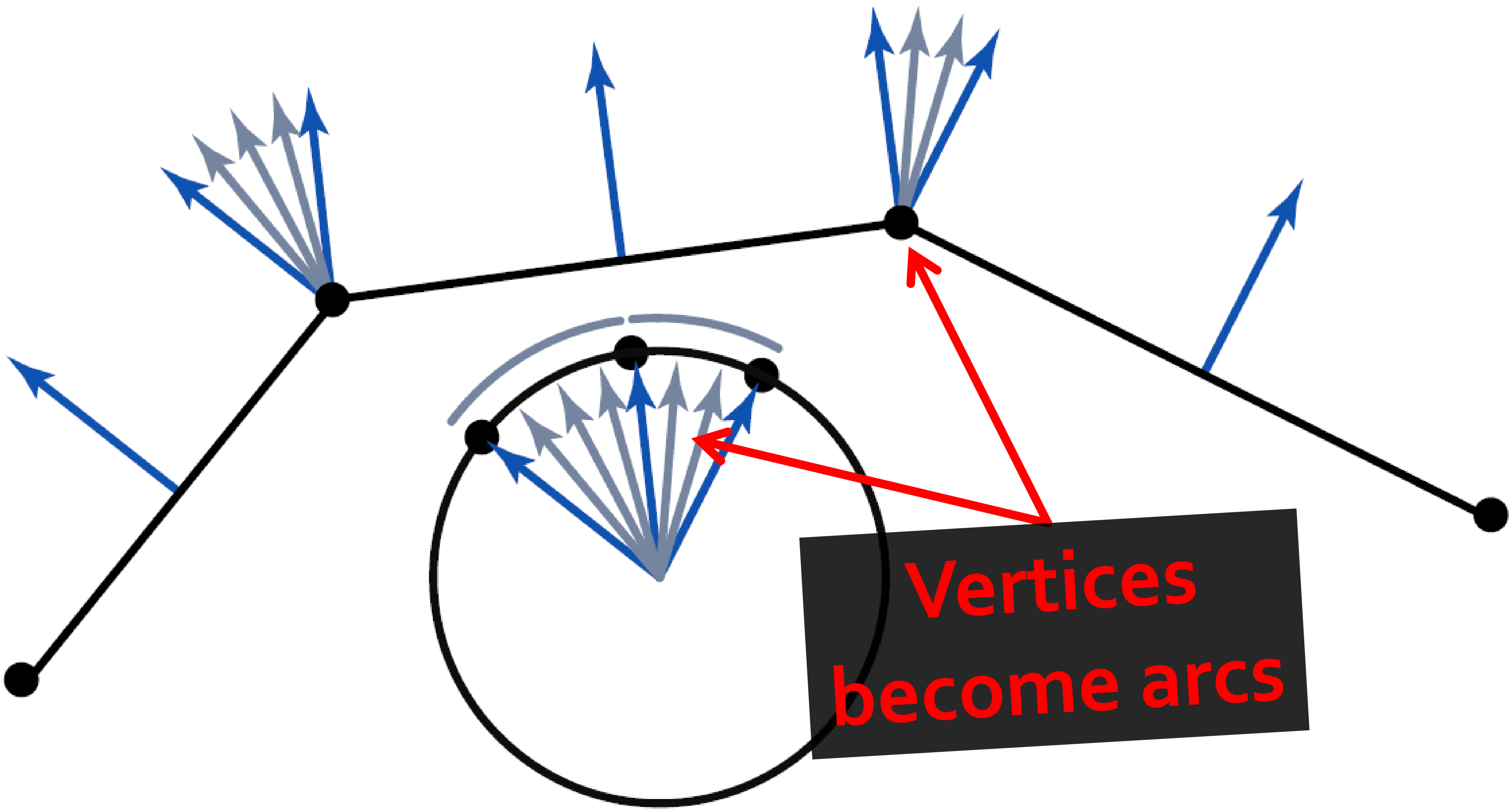
Discrete Gauss Map



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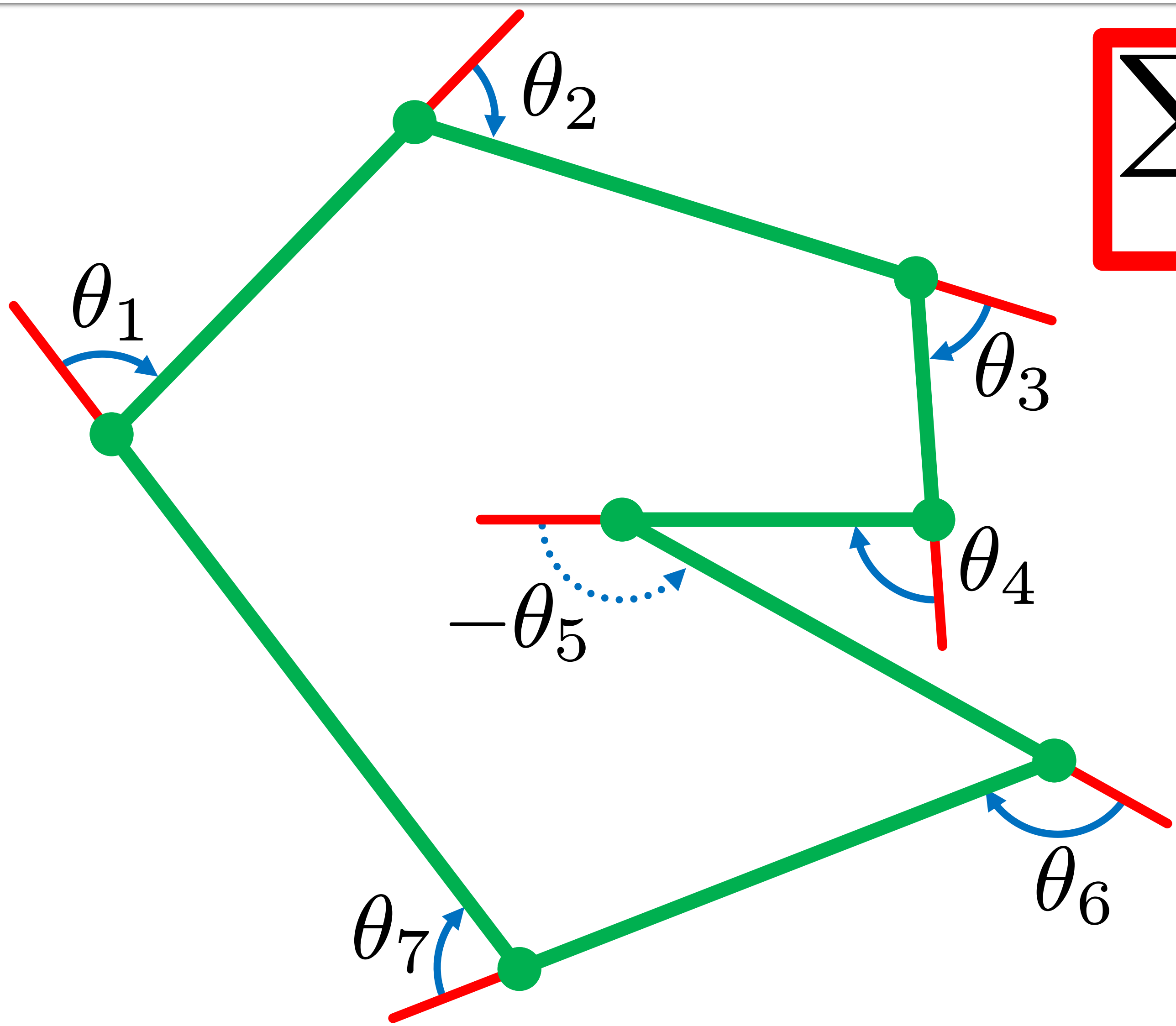
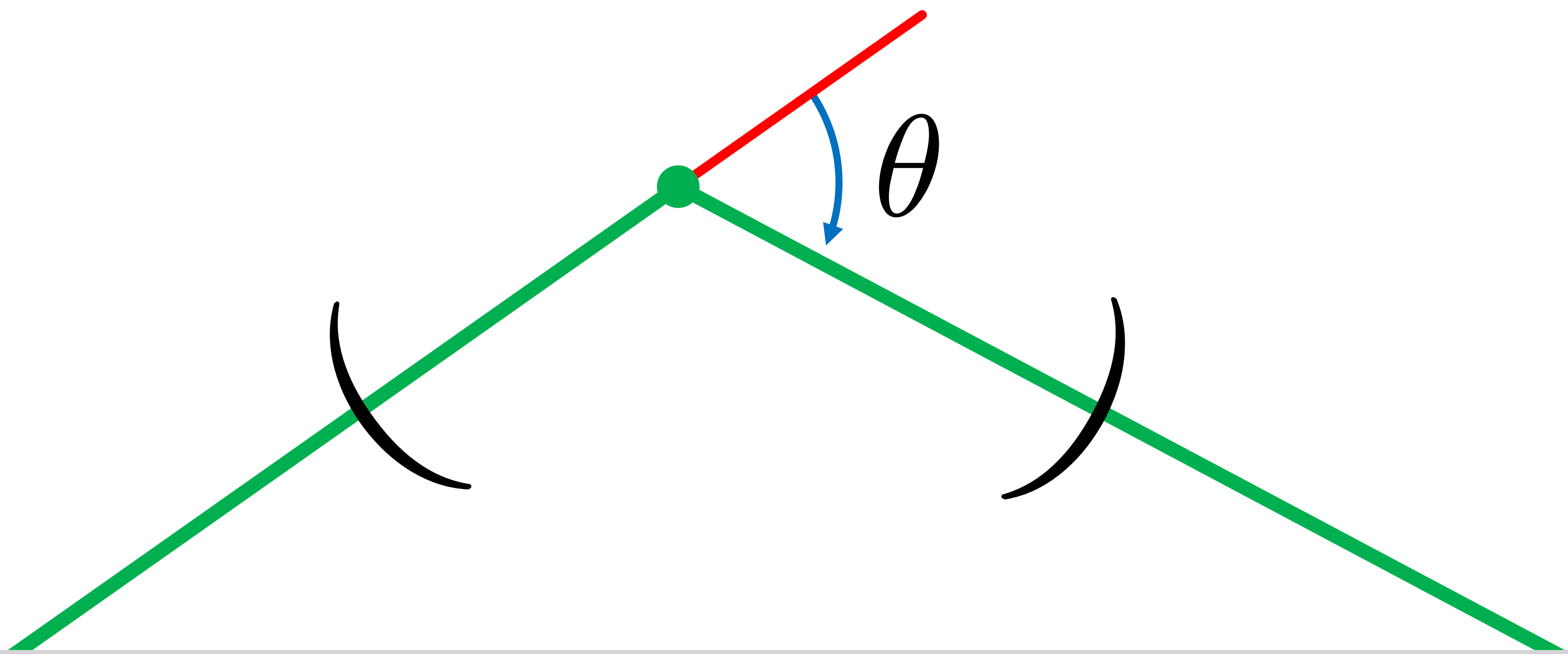
Key Observation



$$\sum_i \theta_i = 2\pi k$$

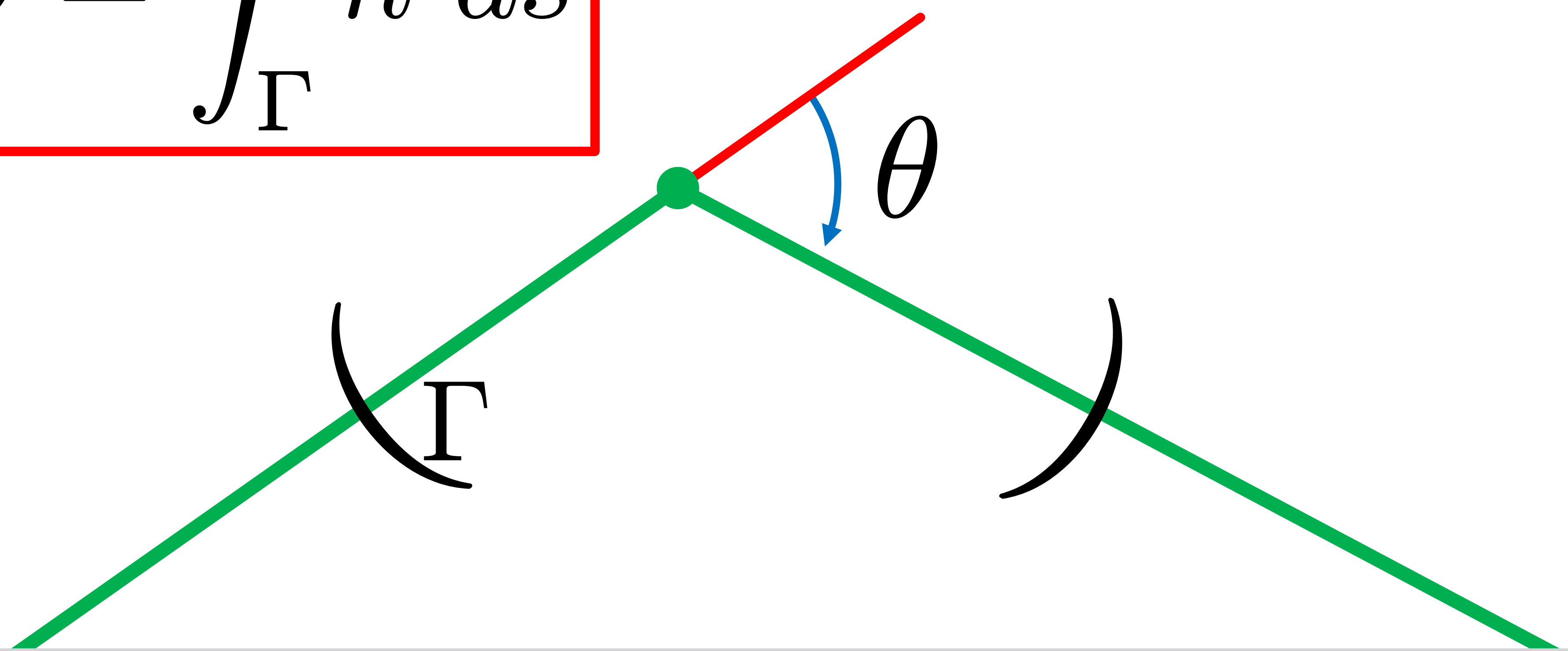
What's Going On?



Total change in curvature

What's Going On?

$$\theta = \int_\Gamma \kappa \; ds$$

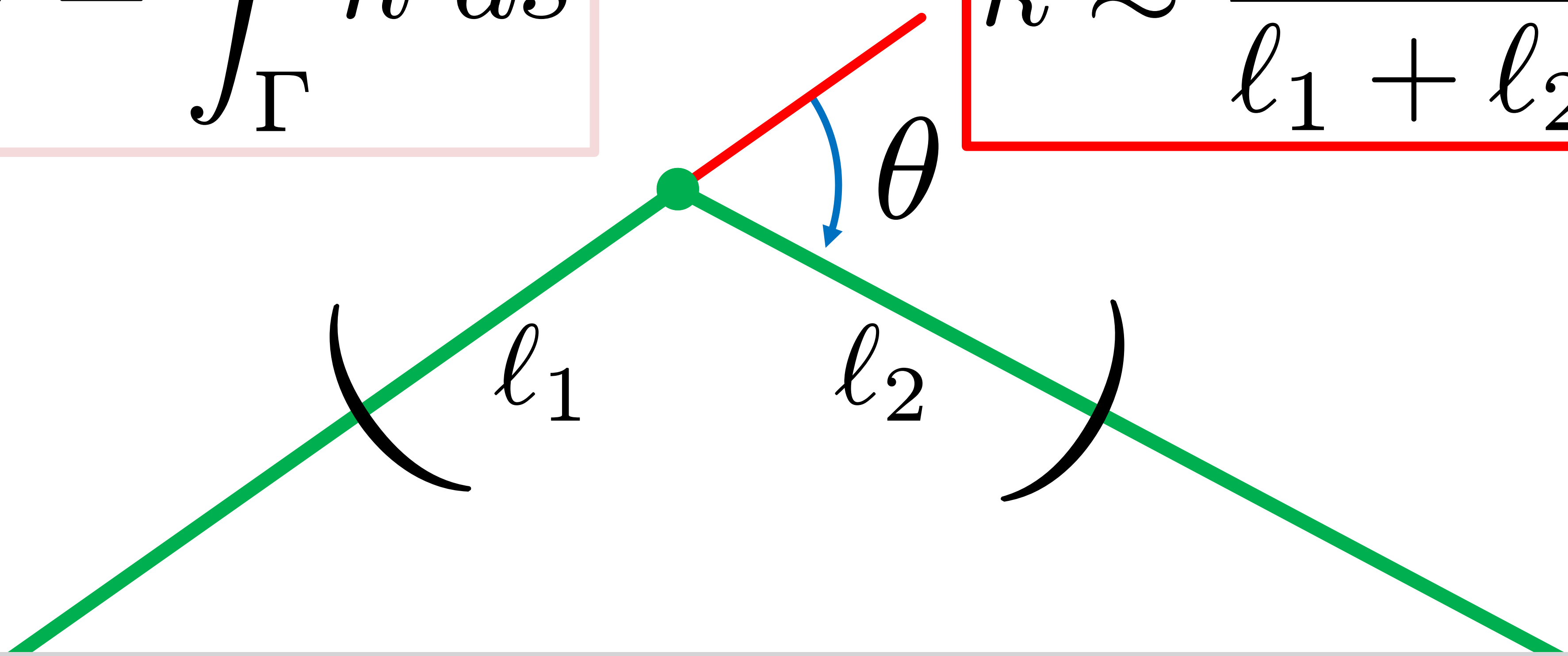


Total change in curvature

What's Going On?

$$\theta = \int_\Gamma \kappa \, ds$$

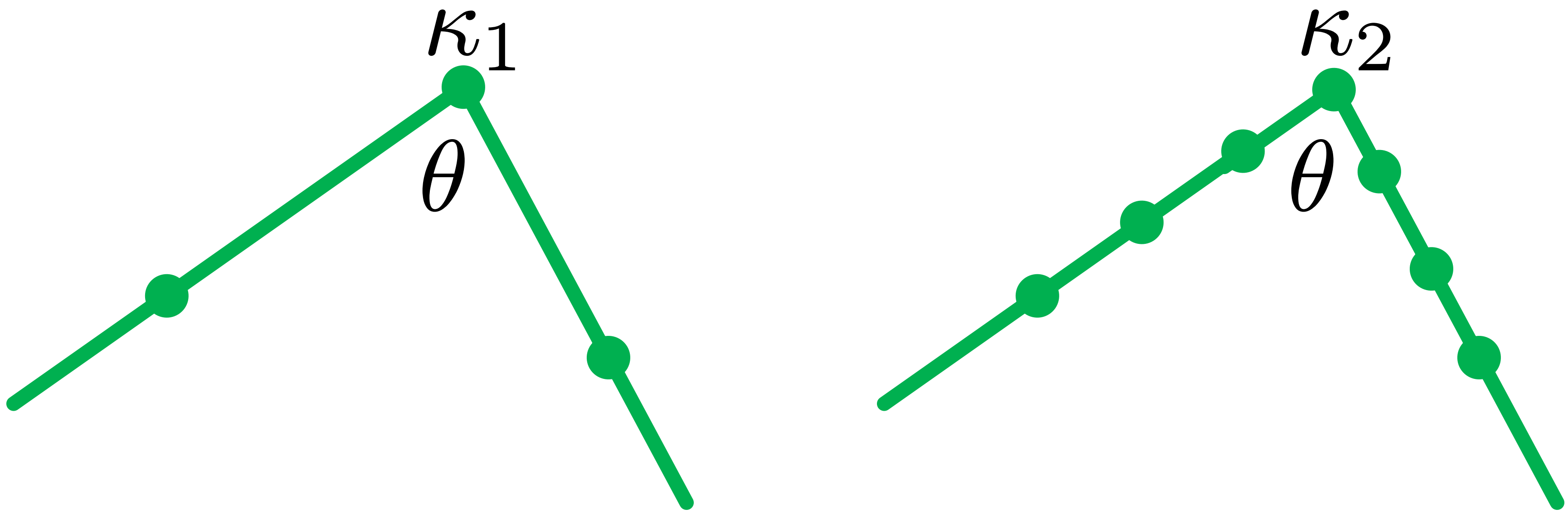
$$\kappa \approx \frac{\theta}{\ell_1 + \ell_2}$$



Total change in curvature

Interesting Distinction

$$\kappa_1 \ll \kappa_2$$



Same integrated curvature

Interesting Distinction

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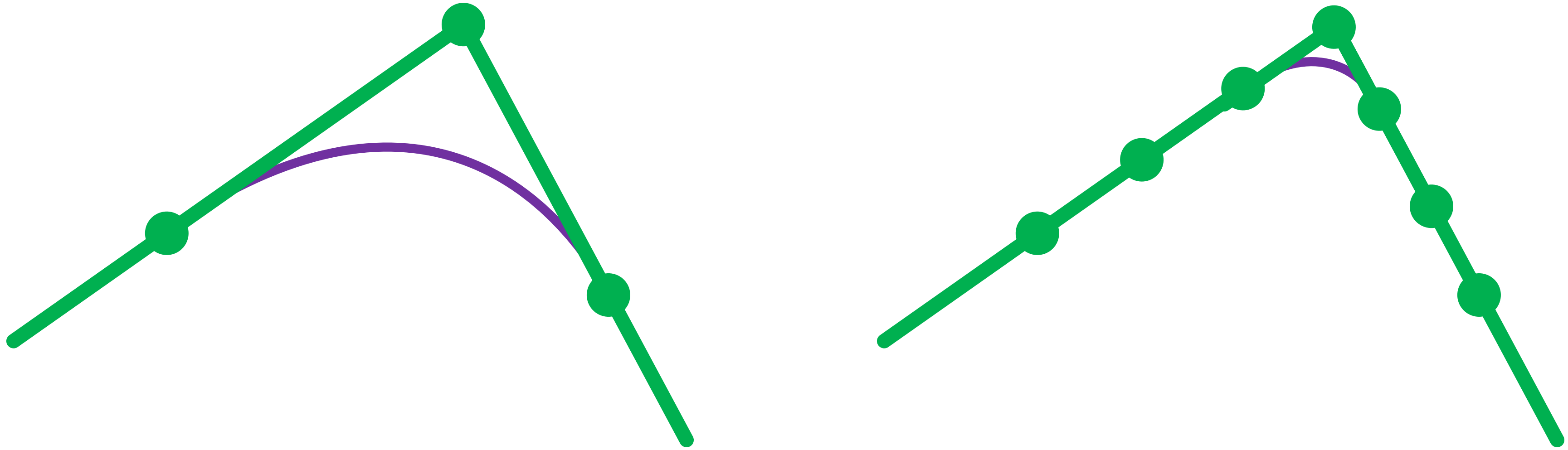
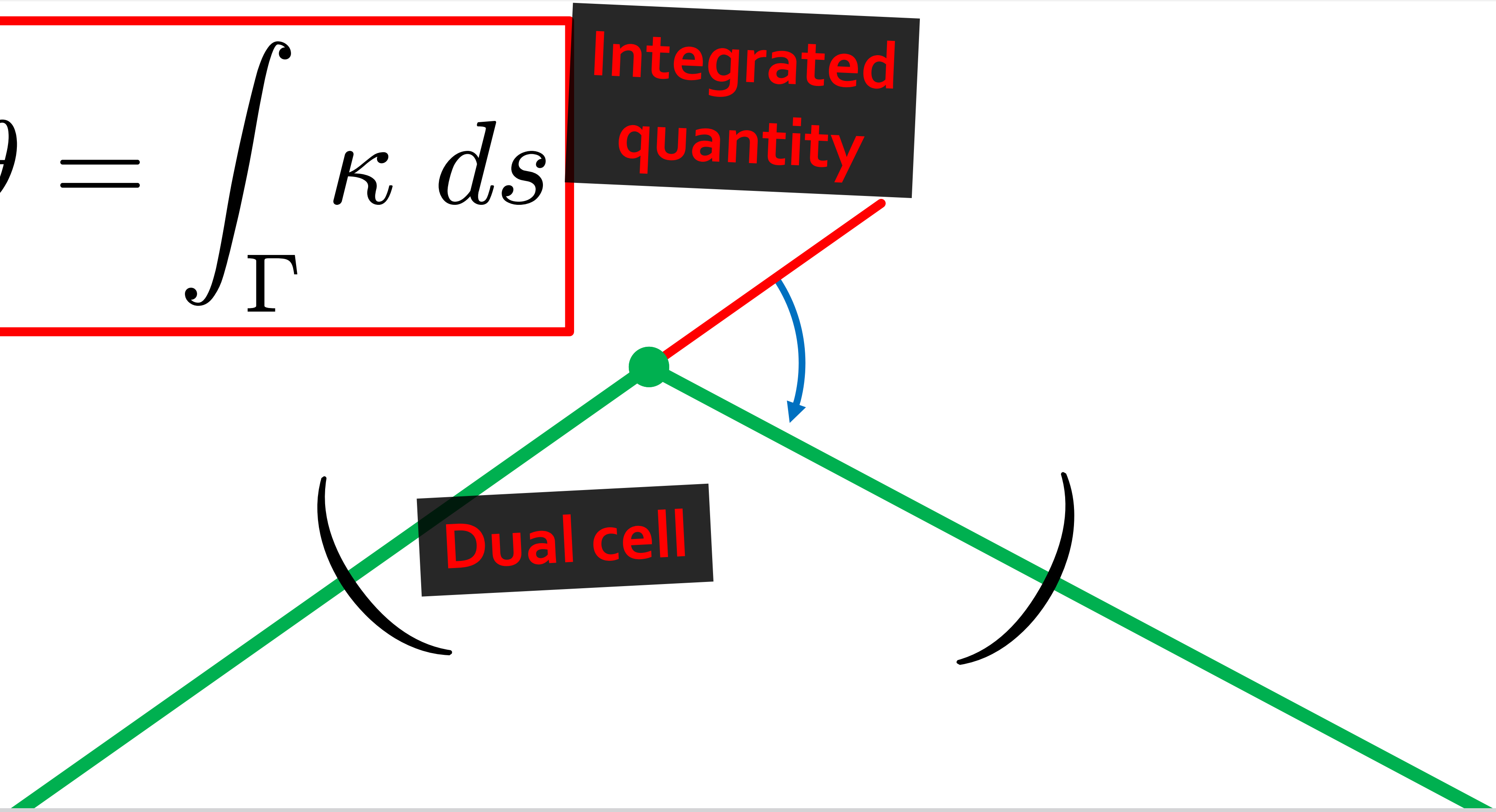


Same integrated curvature

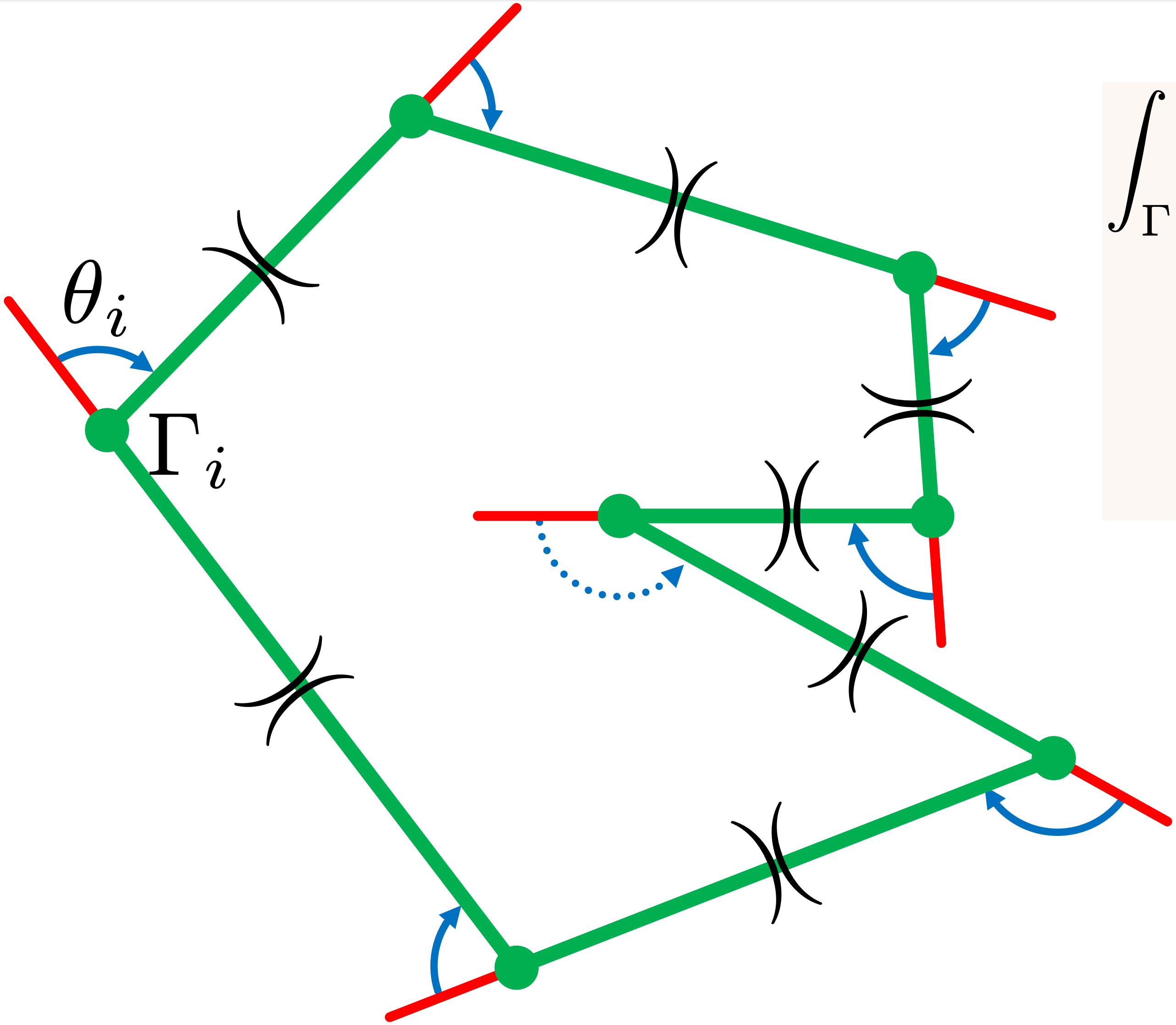
Preview of DEC

$$\theta = \int_\Gamma \kappa \, ds$$



Total change in curvature

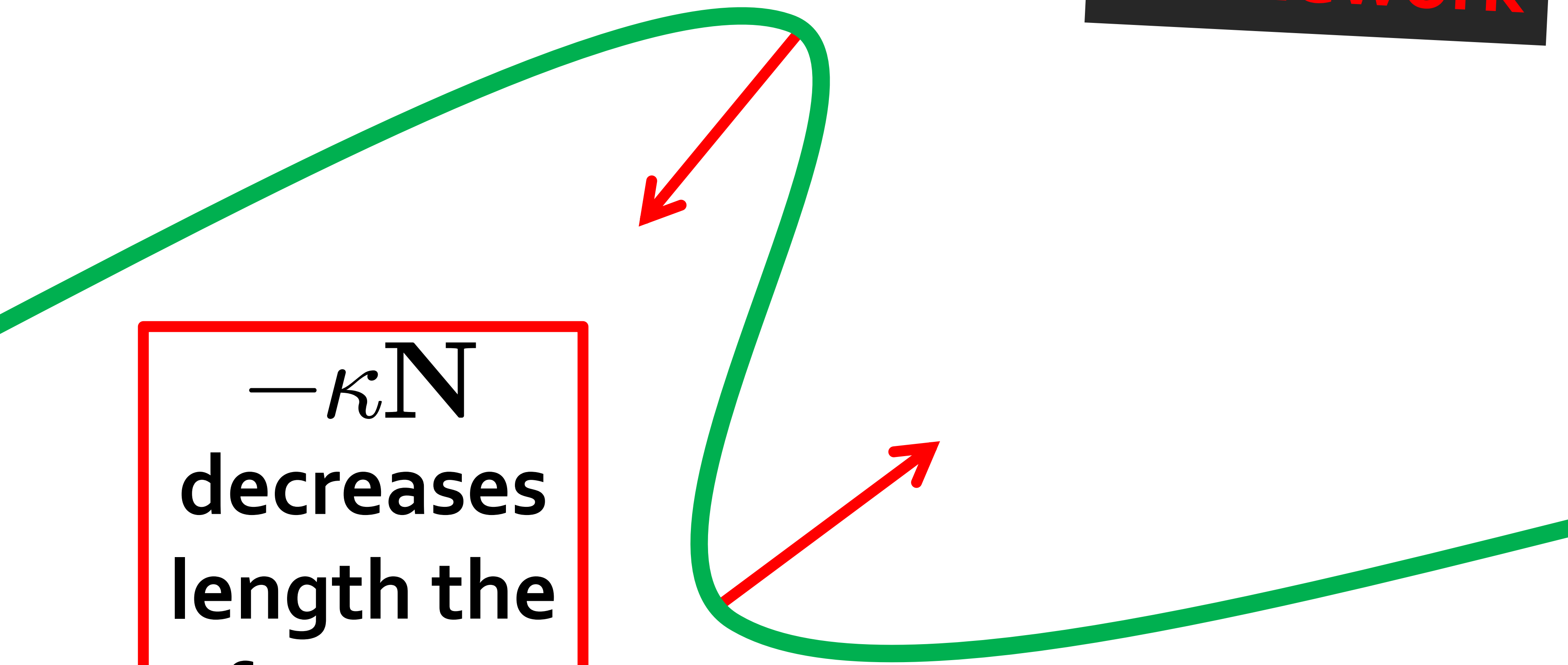
Discrete Turning Angle Theorem



$$\int_\Gamma \kappa \; ds = \sum_i \int_{\Gamma_i} \kappa \; ds$$
$$= \sum_i \theta_i$$
$$= 2\pi k$$

Alternative Definition

Homework



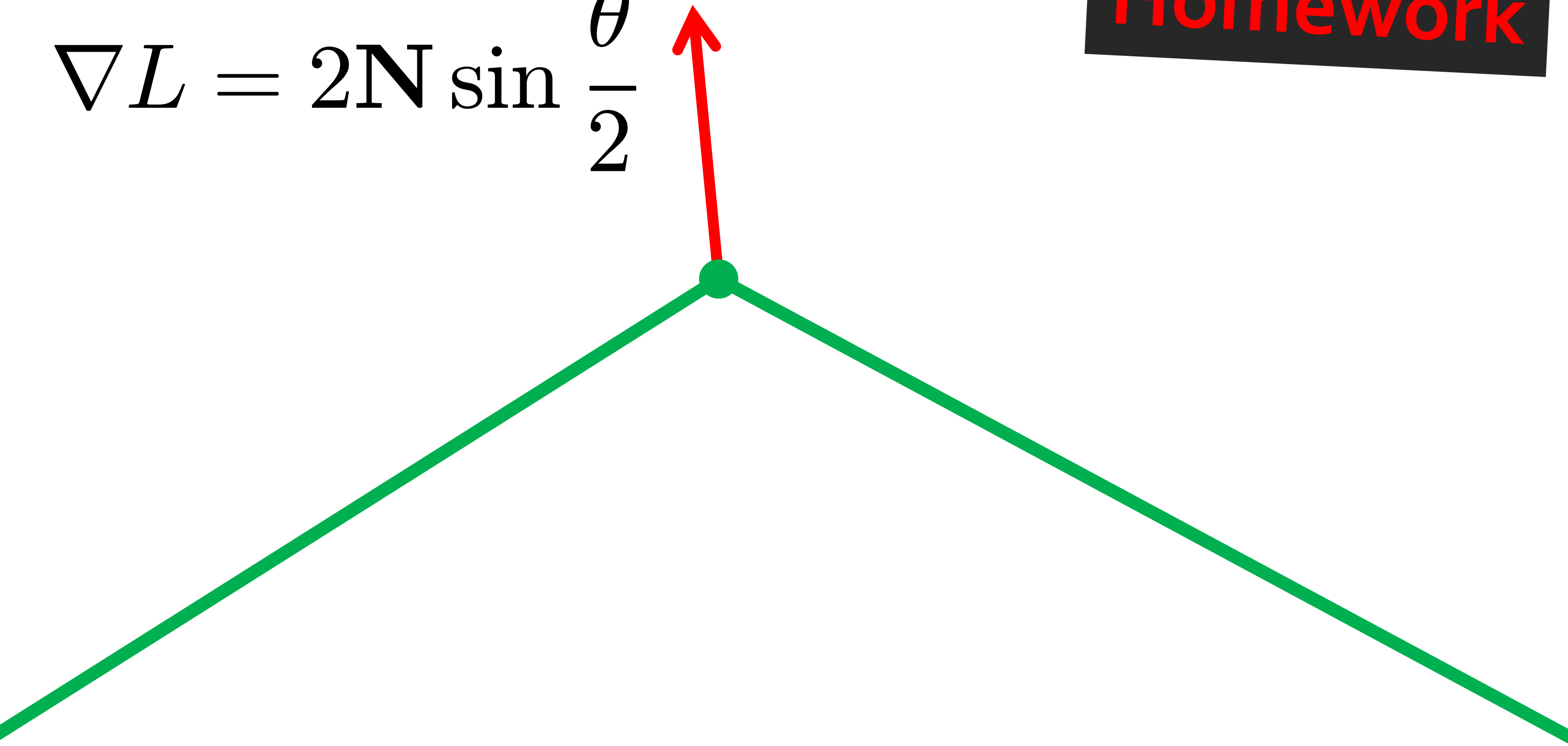
$-\kappa \mathbf{N}$

decreases length the fastest.

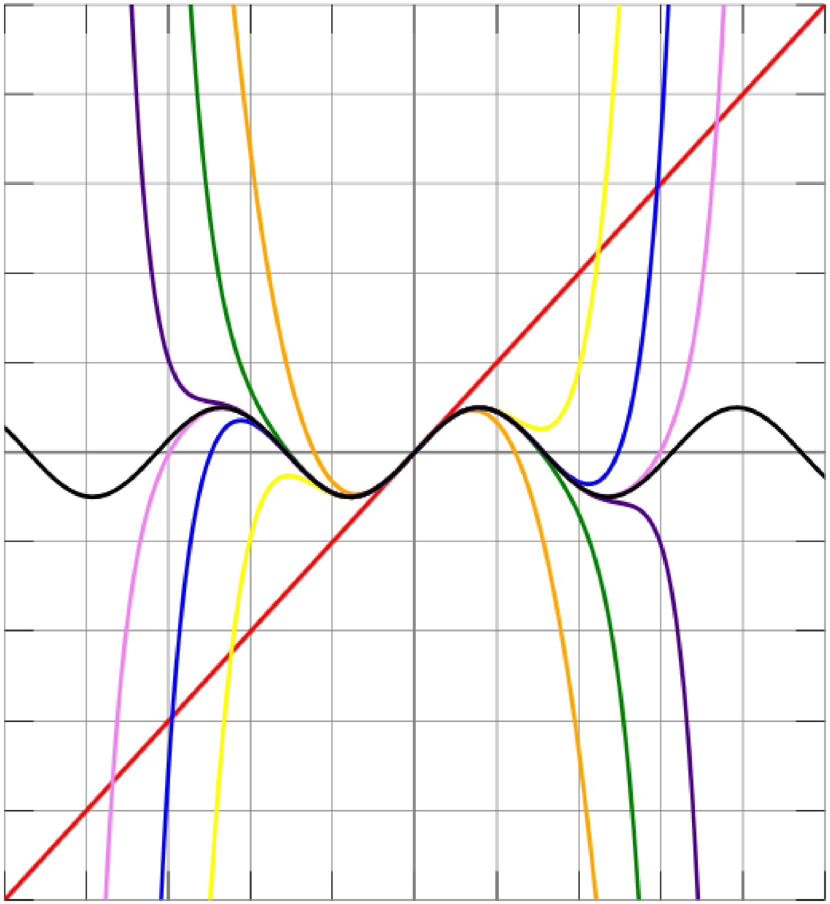
Discrete Case

Homework

$$\nabla L = 2\mathbf{N} \sin \frac{\theta}{2}$$



For Small θ



http://en.wikipedia.org/wiki/Taylor_series

$$2\sin\frac{\theta}{2} \approx 2\cdot\frac{\theta}{2}$$

$$= \theta$$

Same behavior in the limit

Remaining Question

Does discrete curvature converge in limit?

Yes!

Remaining Question

Does discrete curvature converge in limit?

Yes!

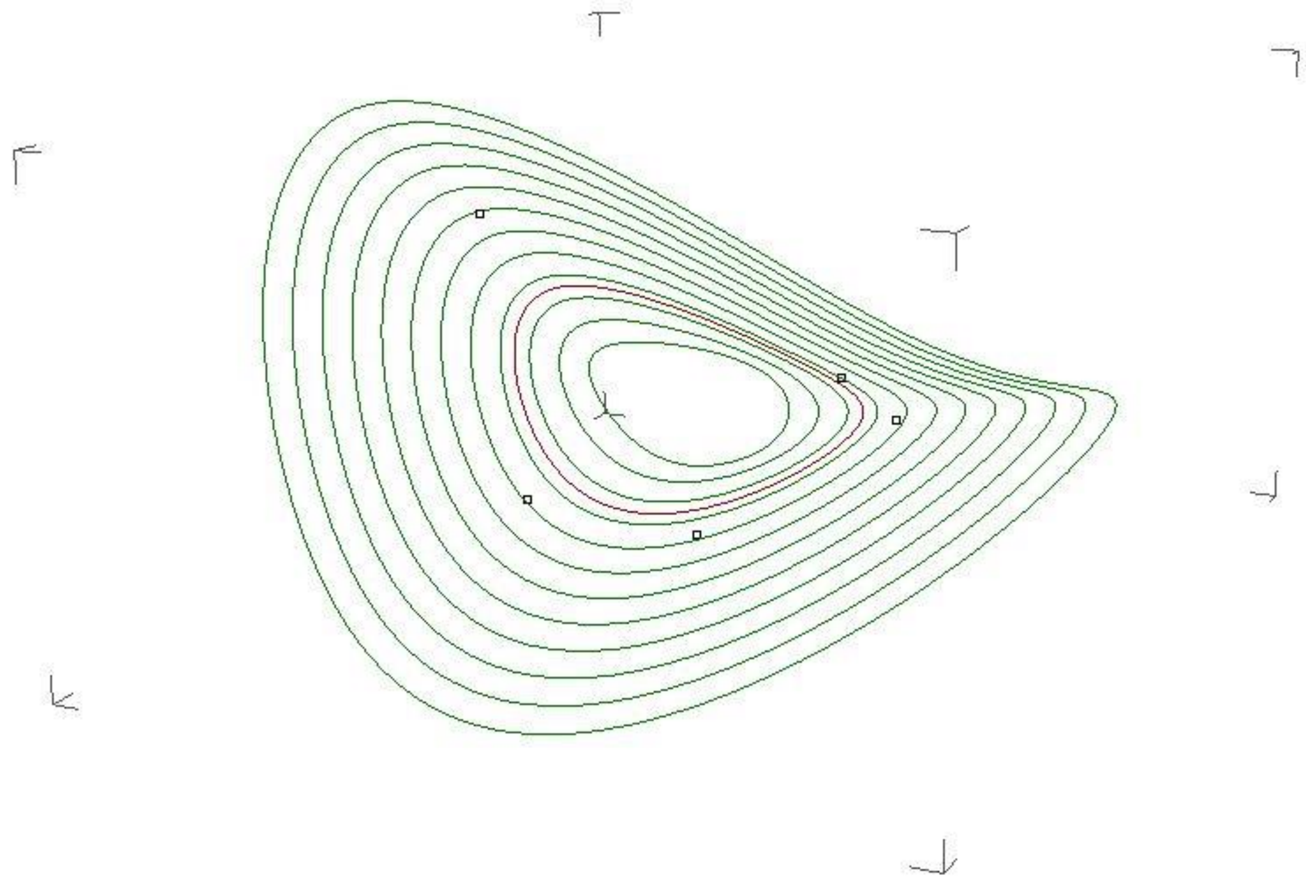
Questions:

- Type of convergence?
- Sampling?
- Class of curves?

Discrete Differential Geometry

- **Different** discrete behavior
- **Same** convergence

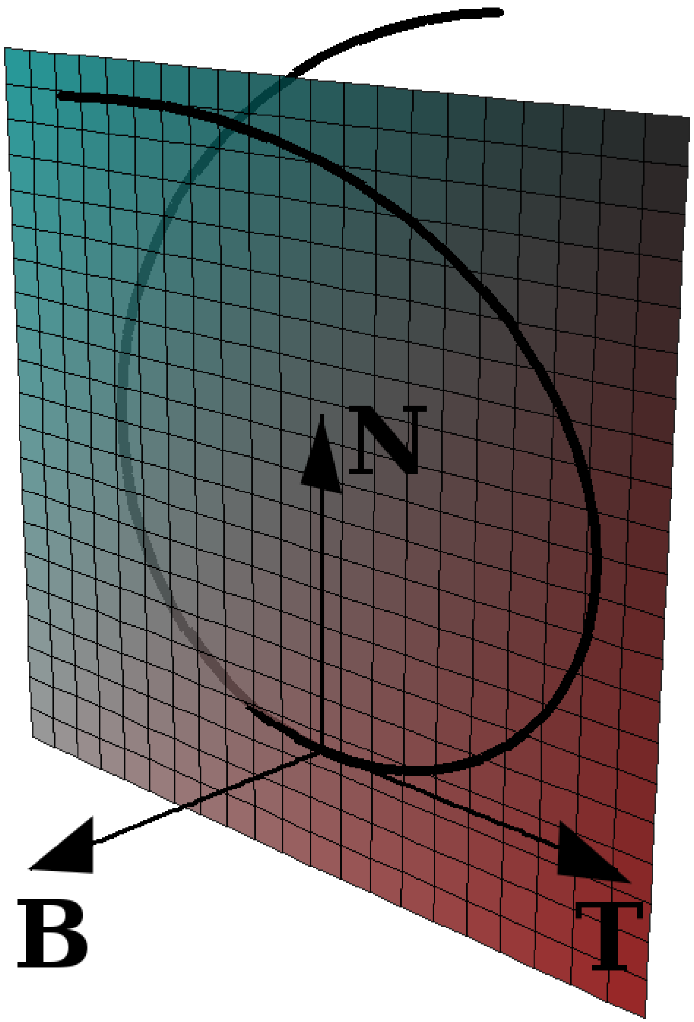
Next



http://www.grasshopper3d.com/forum/topics/offseting-3d-curves-component

Curves in 3D

Frenet Frame



$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

Potential Discretization

$$\mathbf{t}_j = \frac{\mathbf{p}_{j+1} - \mathbf{p}_j}{\|\mathbf{p}_{j+1} - \mathbf{p}_j\|}$$

$$\mathbf{b}_j = \mathbf{t}_{j-1} \times \mathbf{t}_j$$

$$\mathbf{n}_j = \mathbf{b}_j \times \mathbf{t}_j$$

Discrete Frenet frame

$$\mathbf{t}_k = R(\mathbf{b}_k, \theta_k)\mathbf{t}_{k-1}$$

$$\mathbf{b}_{k+1} = R(\mathbf{t}_k, \phi_k)\mathbf{b}_k$$

"Bond and torsion angles"

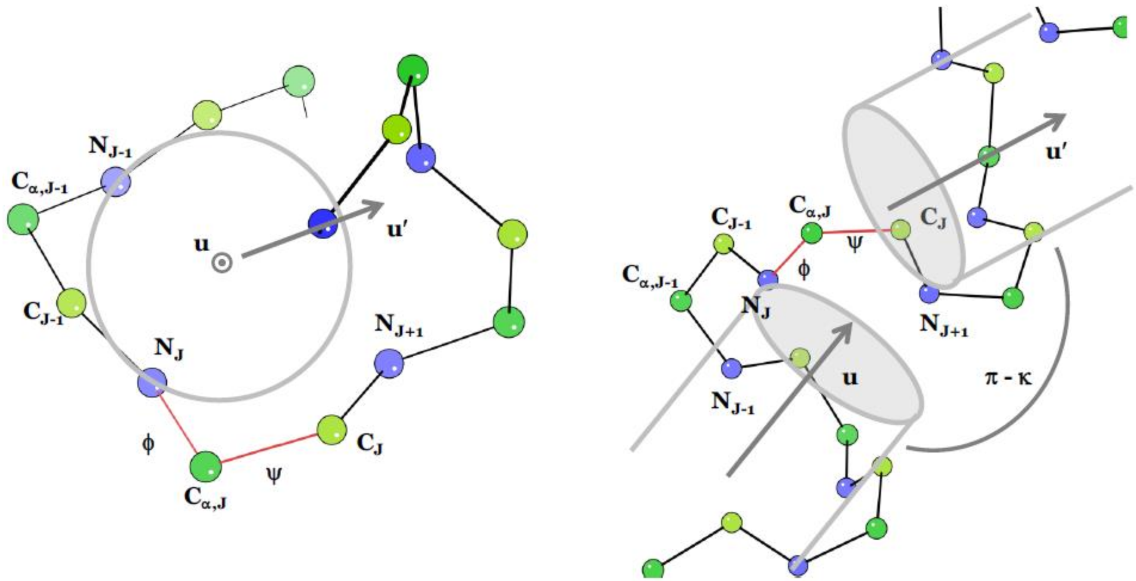
(derivatives converge to κ and τ, resp.)

Discrete frame introduced in:
The resultant electric moment of complex molecules
Eyring, Physical Review, 39(4):746—748, 1932.

Structure Determination of Membrane Proteins Using Discrete Frenet Frames and Solid State NMR Restraints
Achuthan and Quine
Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

Potential Discretization

NMR scanner



Kinked alpha helix

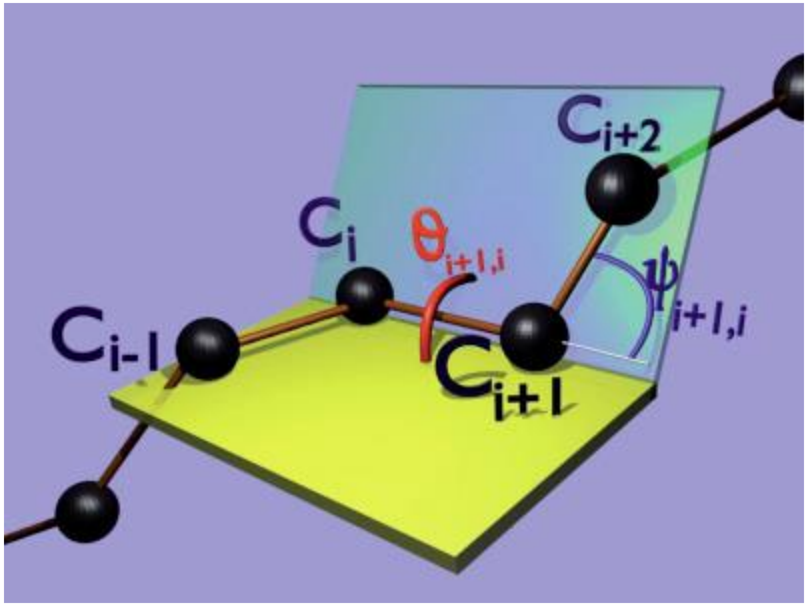
Structure Determination of Membrane Proteins Using Discrete Frenet Frames and Solid State NMR Restraints

Achuthan and Quine

Discrete Mathematics and its Applications, ed. M. Sethumadhavan (2006)

Transfer Matrix

$$\begin{pmatrix} \mathbf{t}_{i+1} \\ \mathbf{n}_{i+1} \\ \mathbf{b}_{i+1} \end{pmatrix} = \mathcal{R}_{i+1,i} \begin{pmatrix} \mathbf{t}_{i} \\ \mathbf{n}_{i} \\ \mathbf{b}_{i} \end{pmatrix}$$



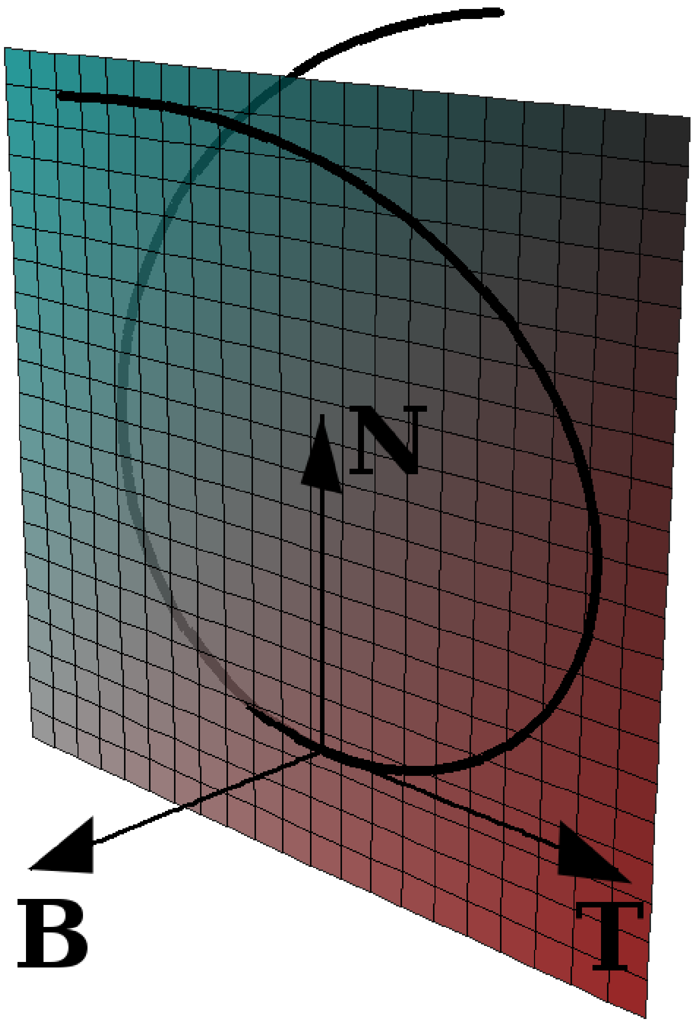
Discrete construction that works for fractal curves and converges in continuum limit.

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins

Hu, Lundgren, and Niemi

Physical Review E 83 (2011)

Frenet Frame: Issue



http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png

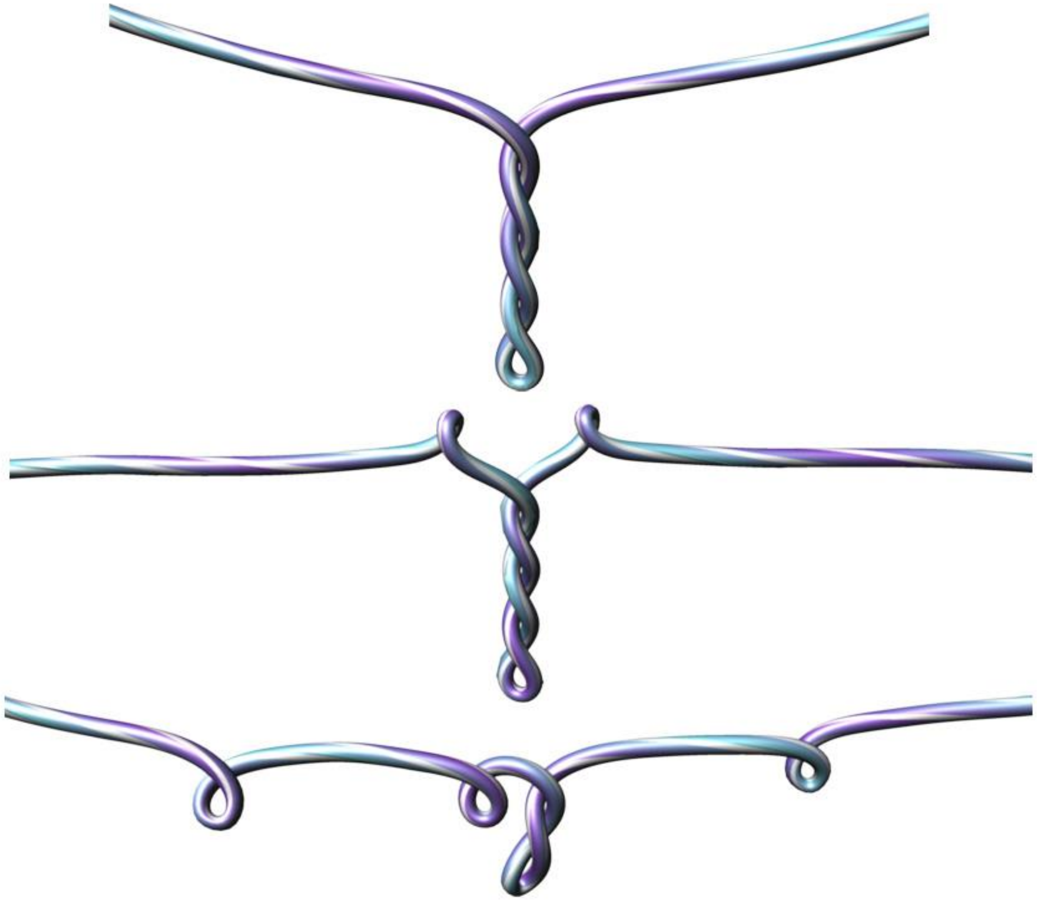
$\kappa = 0$?

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = -\kappa \mathbf{T} + \tau \mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau \mathbf{N}$$

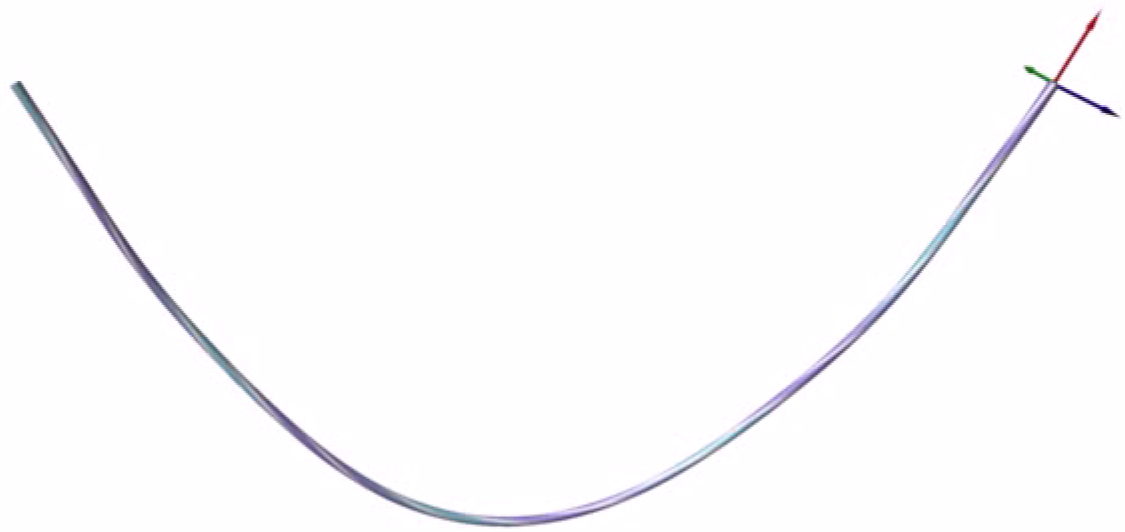
Segments Not Always Enough



Discrete Elastic Rods
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

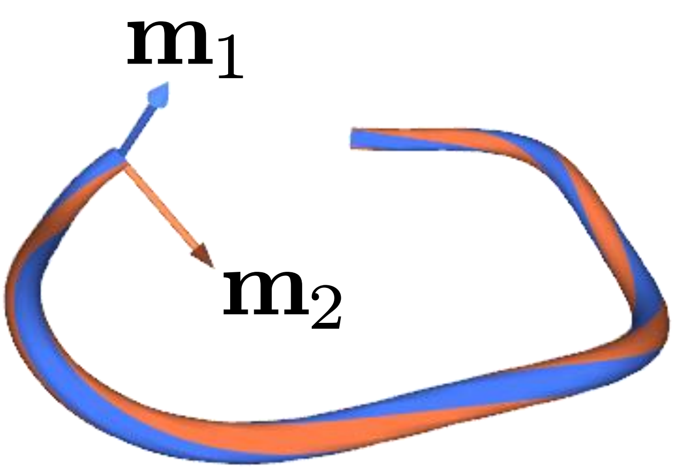
http://www.cs.columbia.edu/cg/rods/

Simulation Goal



http://www.cs.columbia.edu/cg/rods/

Adapted Framed Curve



$$\{\mathbf{t} = \gamma', \mathbf{m}_1, \mathbf{m}_2\}$$

Material frame

http://www.cs.columbia.edu/cg/rods/

Normal part encodes twist

Bending Energy

$$E_{bend}(\Gamma) = \frac{1}{2}\int_\Gamma \alpha\kappa^2 \; ds$$

Punish turning the steering wheel

$$\begin{aligned}\kappa\mathbf{n} &= \mathbf{t}' \\ &= (\mathbf{t}'\cdot\mathbf{t})\mathbf{t} + (\mathbf{t}'\cdot\mathbf{m}_1)\mathbf{m}_1 + (\mathbf{t}'\cdot\mathbf{m}_2)\mathbf{m}_2 \\ &= (\mathbf{t}'\cdot\mathbf{m}_1)\mathbf{m}_1 + (\mathbf{t}'\cdot\mathbf{m}_2)\mathbf{m}_2 \\ &\equiv \omega_1\mathbf{m}_1 + \omega_2\mathbf{m}_2\end{aligned}$$

Bending Energy

$$E_{bend}(\Gamma) = \frac{1}{2}\int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2)\ ds$$

Punish turning the steering wheel

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Twisting Energy

$$E_{twist}(\Gamma) = \frac{1}{2}\int_\Gamma \beta m^2 \; ds$$

Punish non-tangent change in material frame

$$\begin{aligned} m &\equiv \mathbf{m}_1' \cdot \mathbf{m}_2 \\ &= \frac{d}{dt}(\mathbf{m}_1 \cdot \mathbf{m}_2) - \mathbf{m}_1 \cdot \mathbf{m}_2' \\ &= -\mathbf{m}_1 \cdot \mathbf{m}_2' \end{aligned}$$

Twisting Energy

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Swapping m_1 and m_2 does not affect E_{twist}!

Which Basis to Use

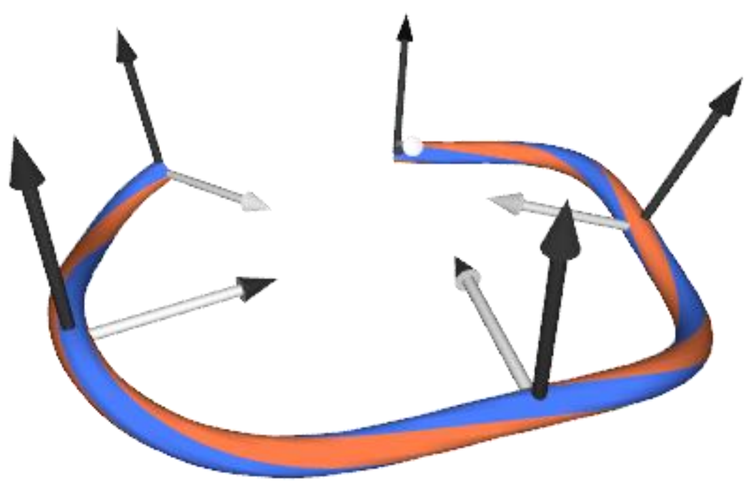
THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, C^3) nondegenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthonormality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is *adapted* to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

1. Relatively parallel fields. We say that a normal vector field M along a curve is *relatively parallel* if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in

Bishop Frame



$$\mathbf{t}' = \quad \times \mathbf{t}$$
$$\mathbf{u}' = \quad \times \mathbf{u}$$
$$\mathbf{v}' = \quad \times \mathbf{v}$$
$$\equiv \kappa \mathbf{b}$$

Darboux vector

http://www.cs.columbia.edu/cg/rods/

Most relaxed frame

Curve-Angle Representation

$$\mathbf{m}_1 = \mathbf{u}\cos\theta + \mathbf{v}\sin\theta$$

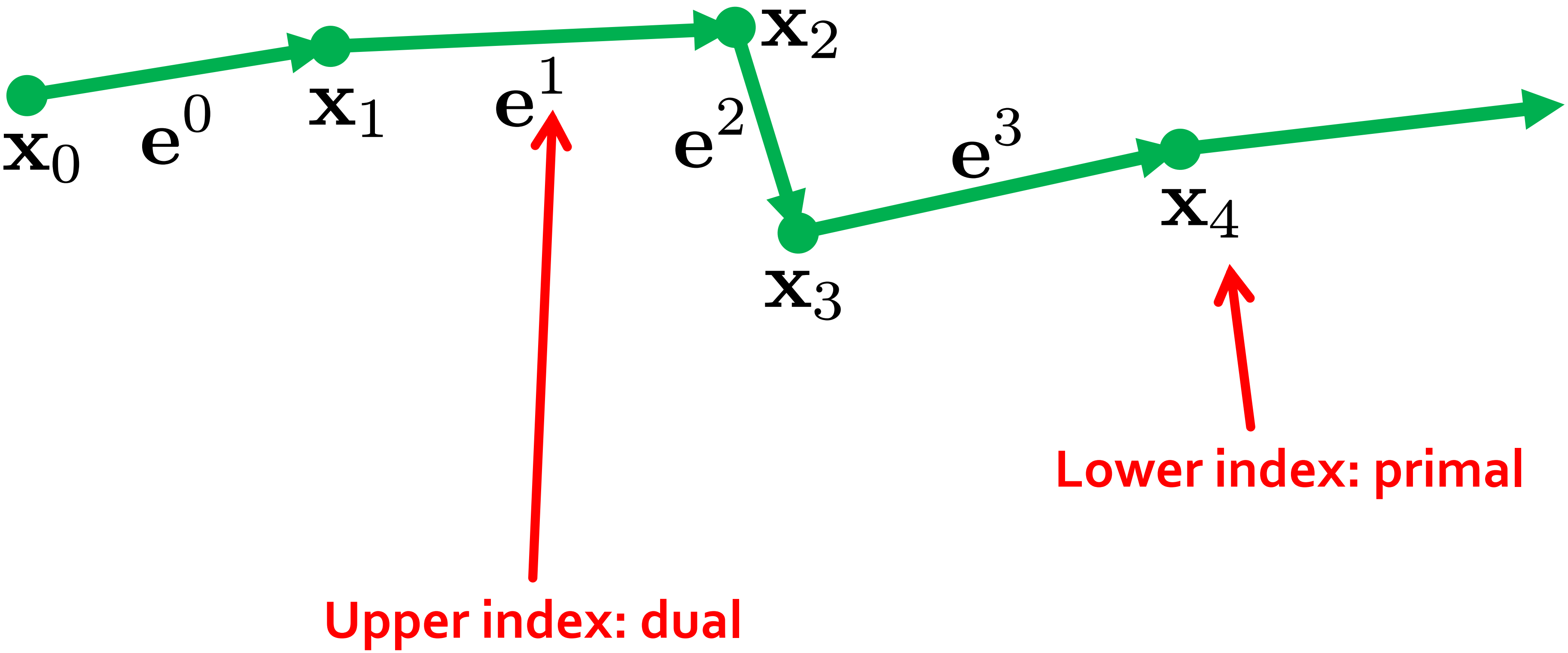
$$\mathbf{m}_2 = -\mathbf{u}\sin\theta + \mathbf{v}\cos\theta$$

$$E_{twist}(\Gamma) = \frac{1}{2}\int_\Gamma \beta(\theta')^2 \; ds$$

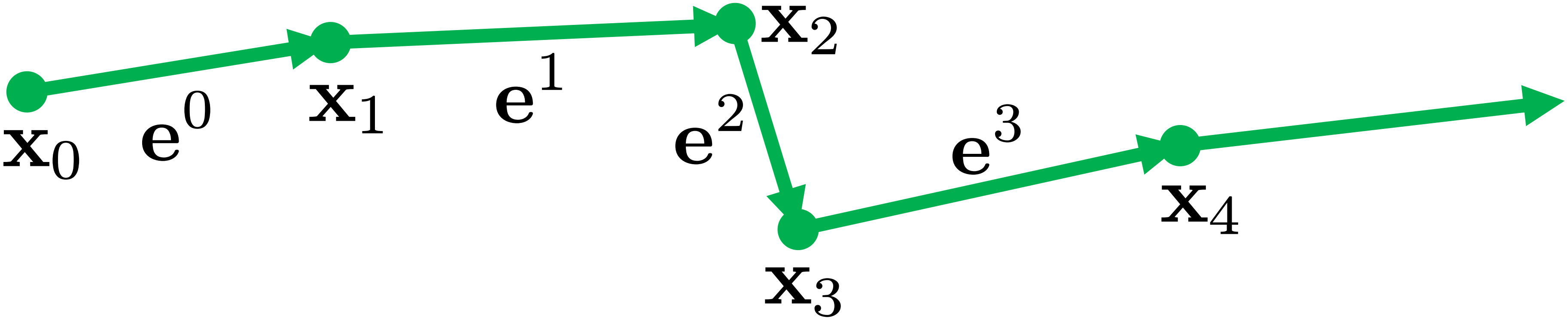
Degrees of freedom for elastic energy:

- **Shape of curve**
- **Twist angle θ**

Discrete Kirchoff Rods



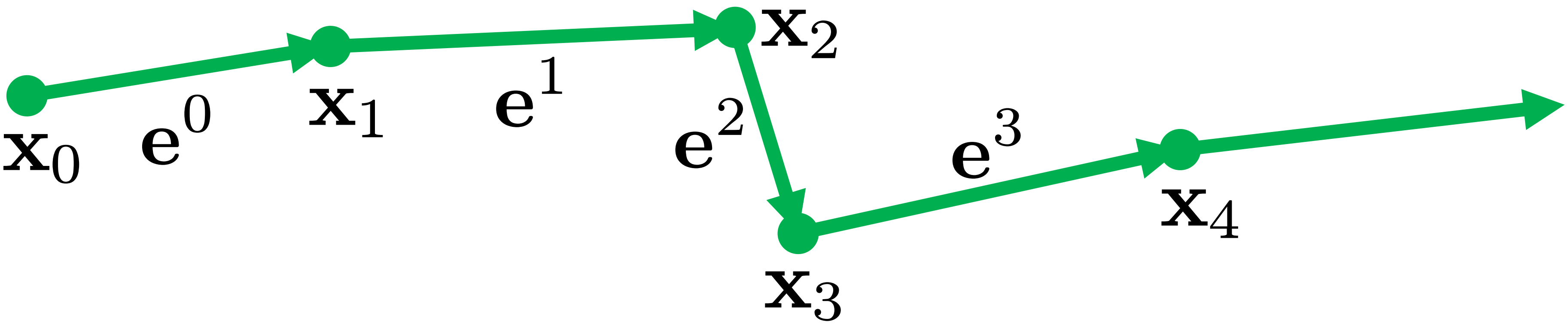
Discrete Kirchoff Rods



$$\mathbf{t}^i = \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|}$$

Tangent unambiguous on edge

Discrete Kirchoff Rods

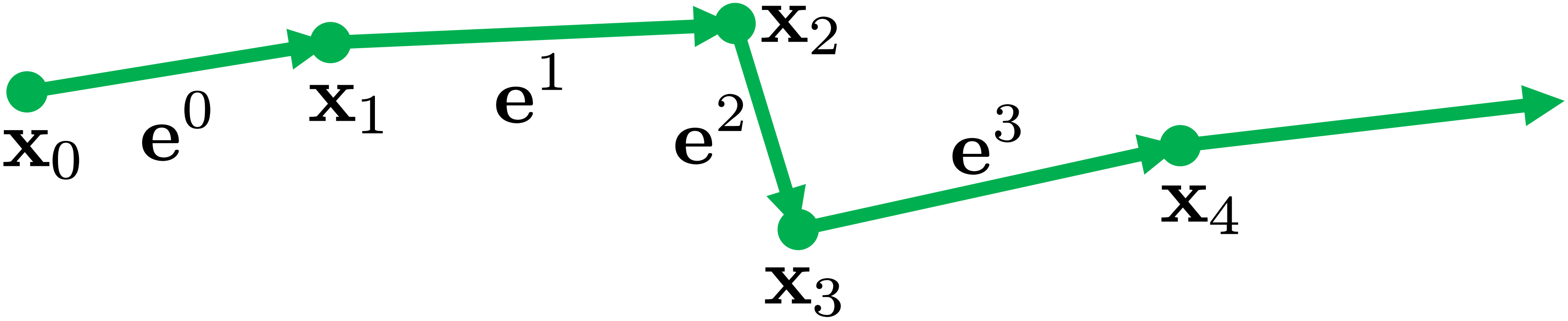


$$\kappa_i = 2 \tan \frac{\phi_i}{2}$$

Yet another curvature!

Integrated curvature

Discrete Kirchoff Rods



$$\kappa_i = 2\tan\frac{\phi_i}{2}$$

Yet another curvature!

$$(\kappa \mathbf{b})_i = \frac{2\mathbf{e}^{i-1} \times \mathbf{e}^i}{\|\mathbf{e}^{i-1}\|\|\mathbf{e}^i\| + \mathbf{e}^{i-1} \cdot \mathbf{e}^i}$$

Orthogonal to osculating plane, norm κ_i

Darboux vector

Bending Energy

$$E_{bend}(\Gamma) = \frac{\alpha}{2}\sum_i \left(\frac{(\kappa\mathbf{b})_i}{\ell_i/2}\right)^2 \frac{\ell_i}{2}$$

$$= \alpha \sum_i \frac{\|(\kappa\mathbf{b})_i\|^2}{\ell_i}$$

Can extend for natural bend

Convert to pointwise and integrate

Discrete Parallel Transport

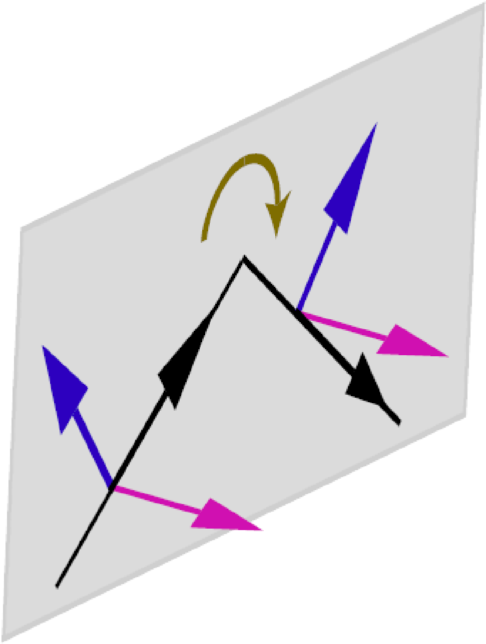
$$P_i(\mathbf{t}^{i-1}) = \mathbf{t}^i$$

$$P_i(\mathbf{t}^{i-1} \times \mathbf{t}^i) = \mathbf{t}^{i-1} \times \mathbf{t}^i$$

- **Map tangent to tangent**
- **Preserve binormal**
- **Orthogonal**

$$\mathbf{u}^i = P_i(\mathbf{u}^{i-1})$$

$$\mathbf{v}^i = \mathbf{t}^i \times \mathbf{u}^i$$

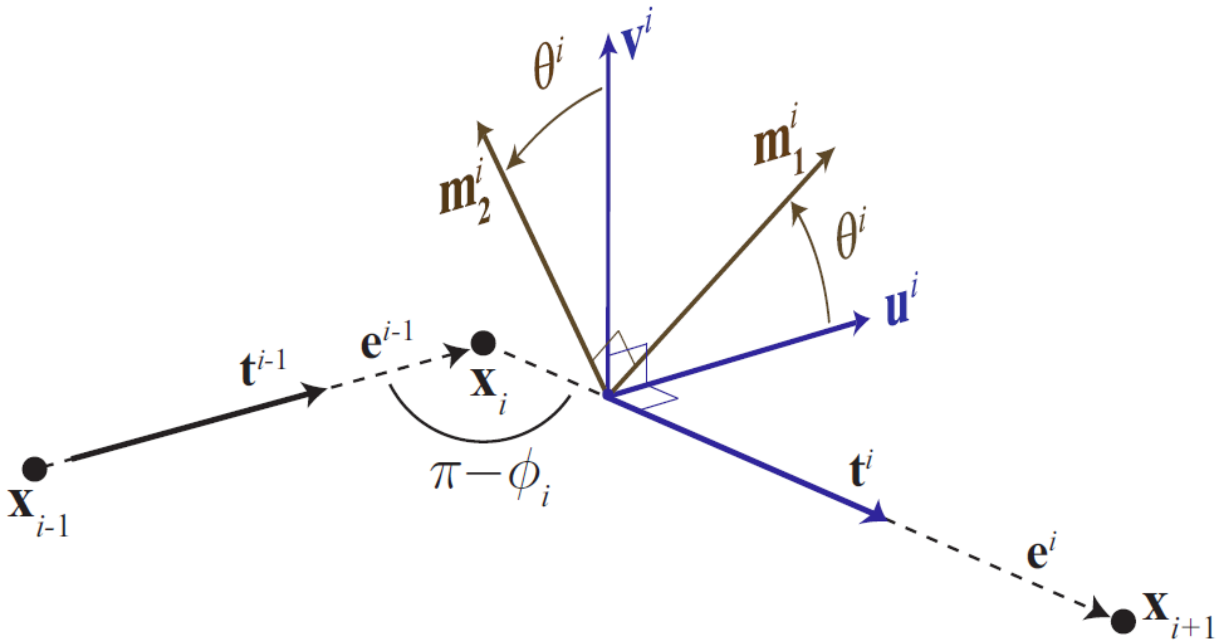


http://www.cs.columbia.edu/cg/rods/

Discrete Material Frame

$$\mathbf{m}_1^i = \mathbf{u}^i \cos\theta^i + \mathbf{v}^i \sin\theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin\theta^i + \mathbf{v}^i \cos\theta^i$$



http://www.cs.columbia.edu/cg/rods/

Discrete Twisting Energy

$$E_{twist}(\Gamma) = \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$

Discrete Twisting Energy

$$E_{twist}(\Gamma) = \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i}$$

Note θ_0 can be arbitrary

Simulation

```
\omit{physics}
```

Simulation

`\omit{physics}`

Worth reading!

Extension and Speedup

Discrete Viscous Threads

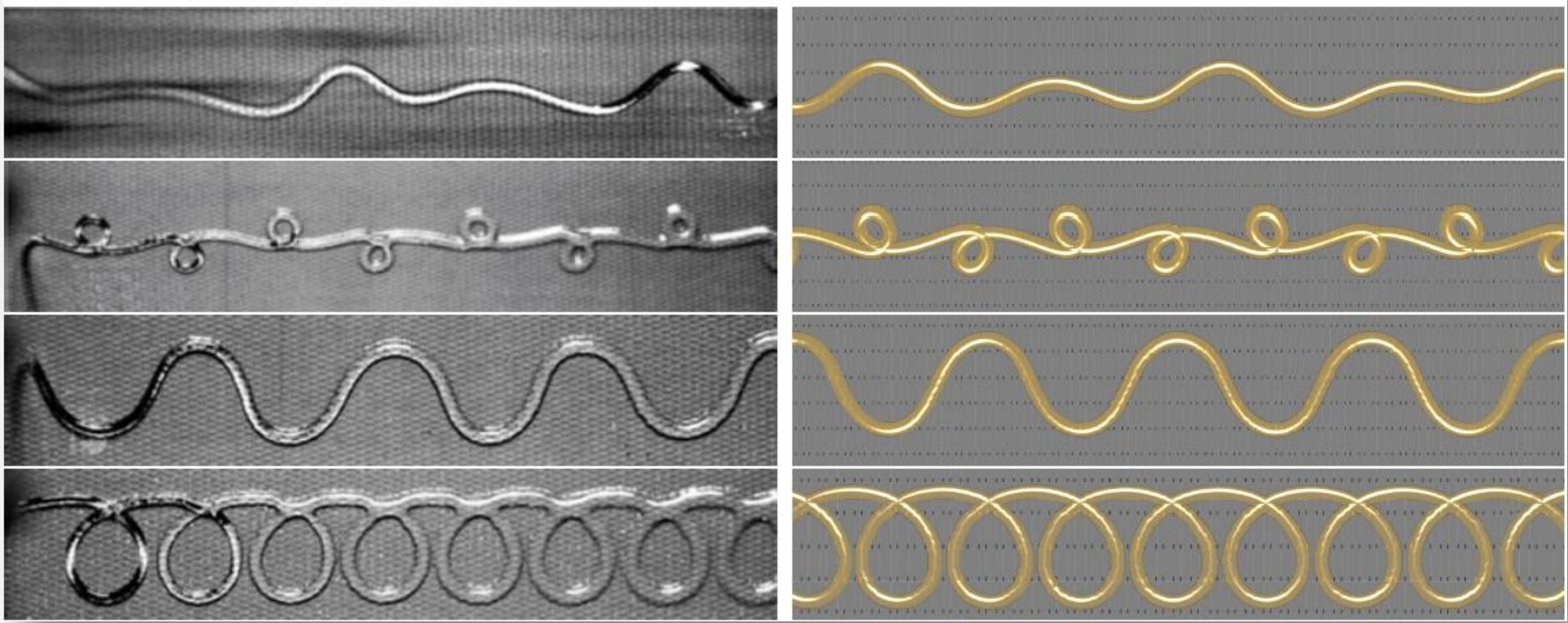
Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

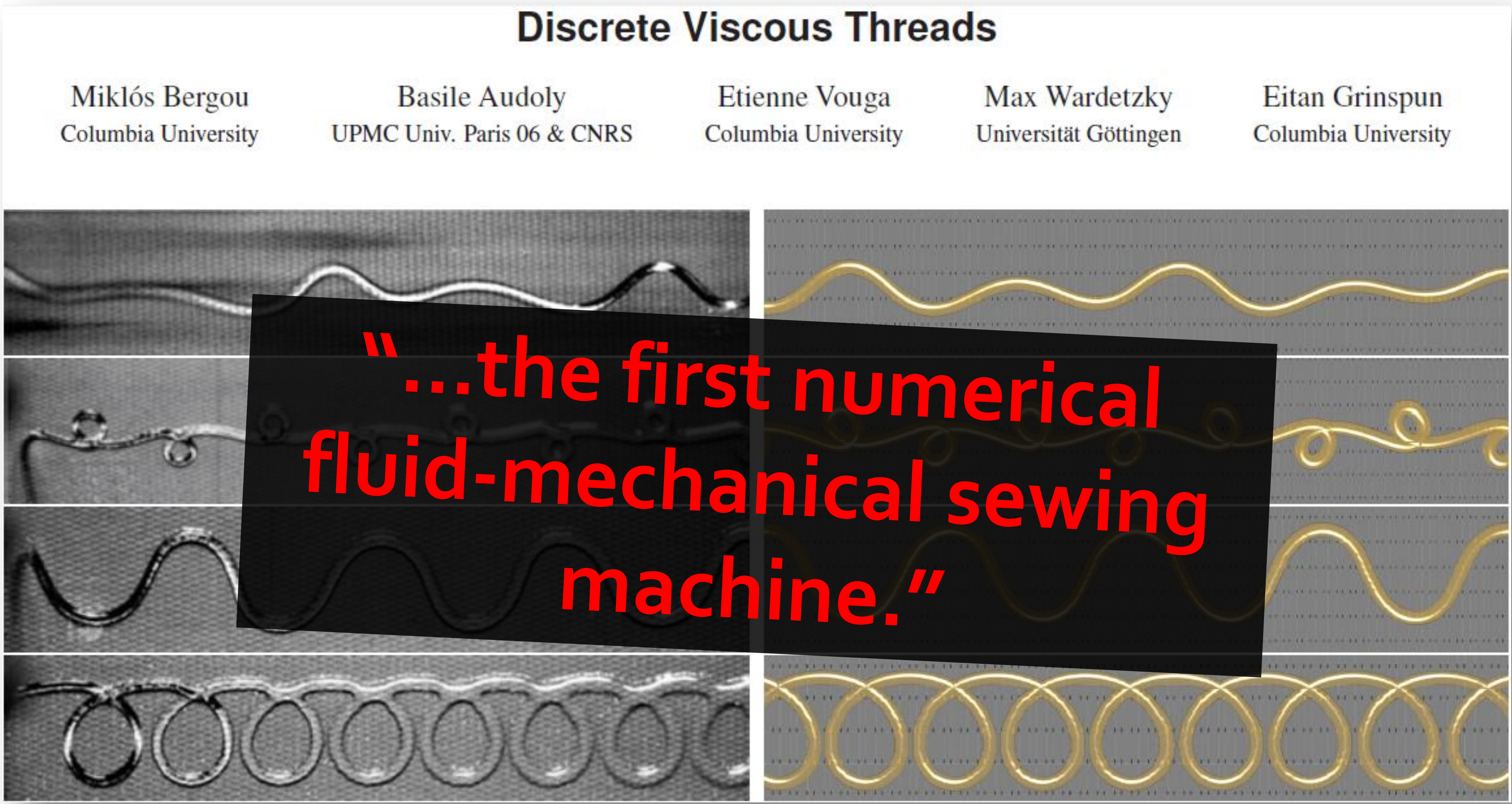
Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University



http://www.cs.columbia.edu/cg/threads/

Extension and Speedup



"…the first numerical fluid-mechanical sewing machine."

http://www.cs.columbia.edu/cg/threads/

Morals

One curve,
three curvatures.

$$\theta \qquad 2\sin\frac{\theta}{2} \qquad 2\tan\frac{\theta}{2}$$

Morals

Easy theoretical object, hard to use.

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

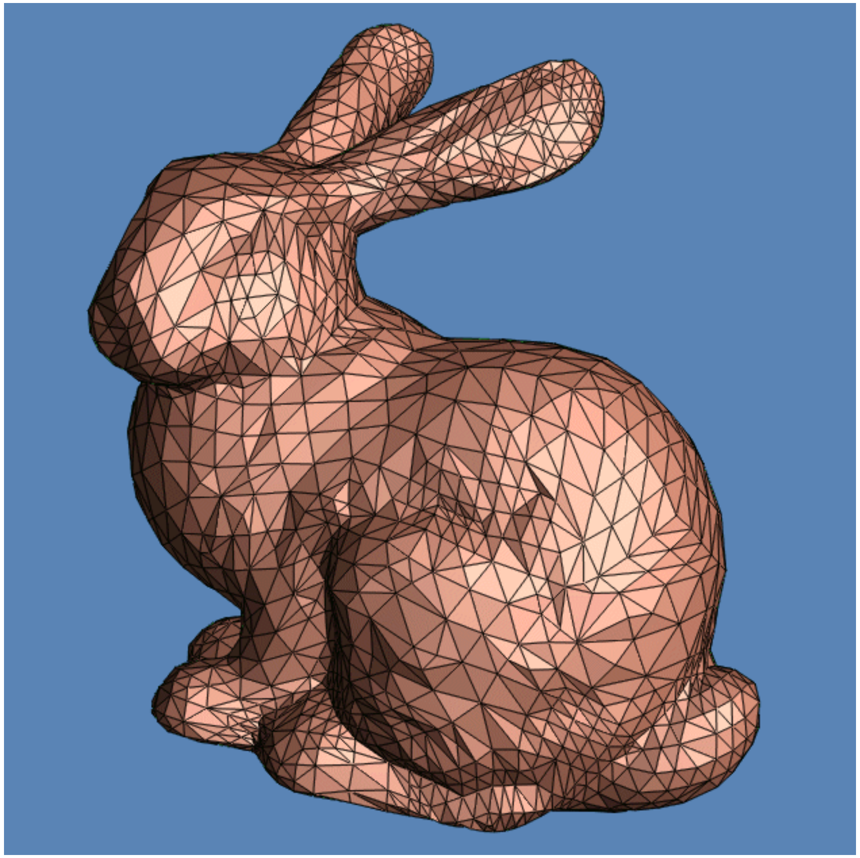
Morals

Proper coordinates and DOFs go a long way.

$$\mathbf{m}_1^i = \mathbf{u}^i \cos\theta^i + \mathbf{v}^i \sin\theta^i$$

$$\mathbf{m}_2^i = -\mathbf{u}^i \sin\theta^i + \mathbf{v}^i \cos\theta^i$$

Next



http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif

Surfaces

http://www.alvinomassage.com/images/knot.jpg

Discrete Curves

CS 468, Spring 2013

Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher