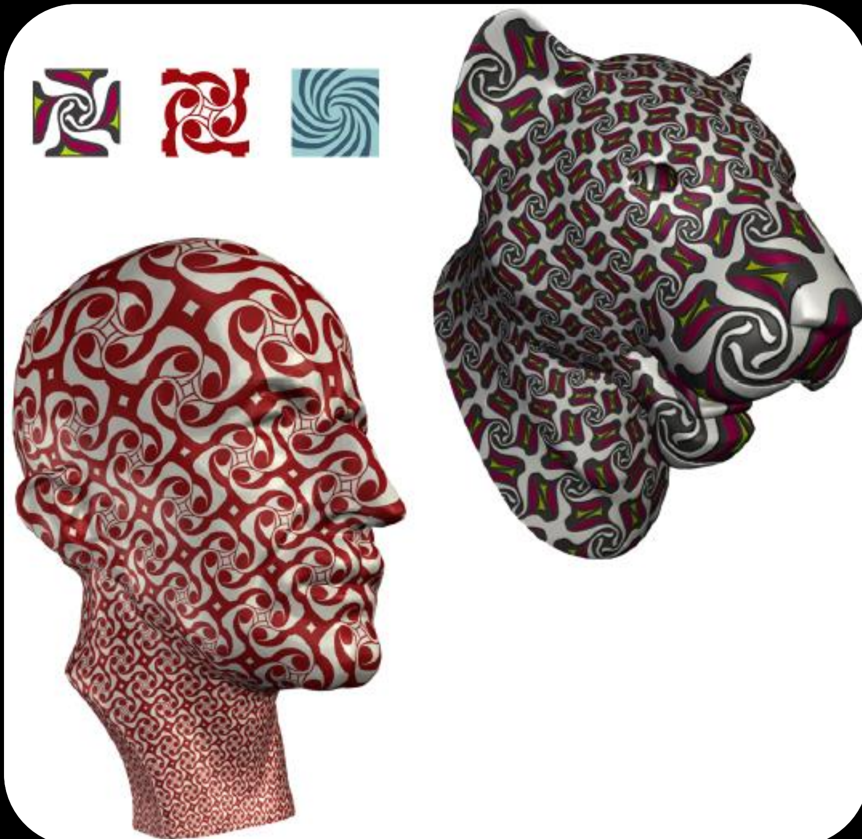
http://www.multires.caltech.edu/pubs/ConfEquiv.pdf http://farm4.static.flickr.com/3145/2556553386_2826612607.jpg

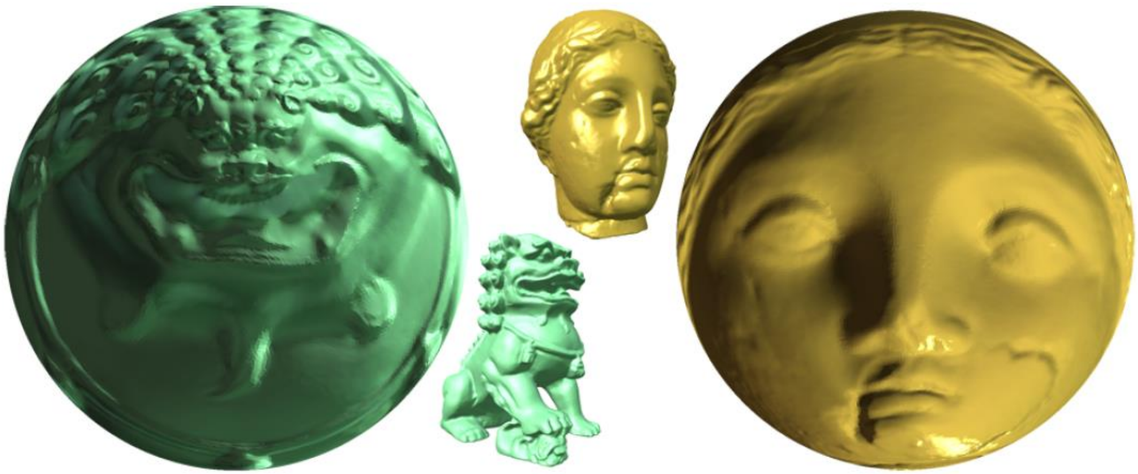


Conformal Methods: Examples

CS 468, Spring 2013
Differential Geometry for Computer Science

Justin Solomon and Adrian Butscher

Example Problem



http://www.multires.caltech.edu/pubs/ConfEquiv.pdf
http://onlinelibrary.wiley.com/doi/10.1111/j.1467-8659.2008.01142.x/abstract

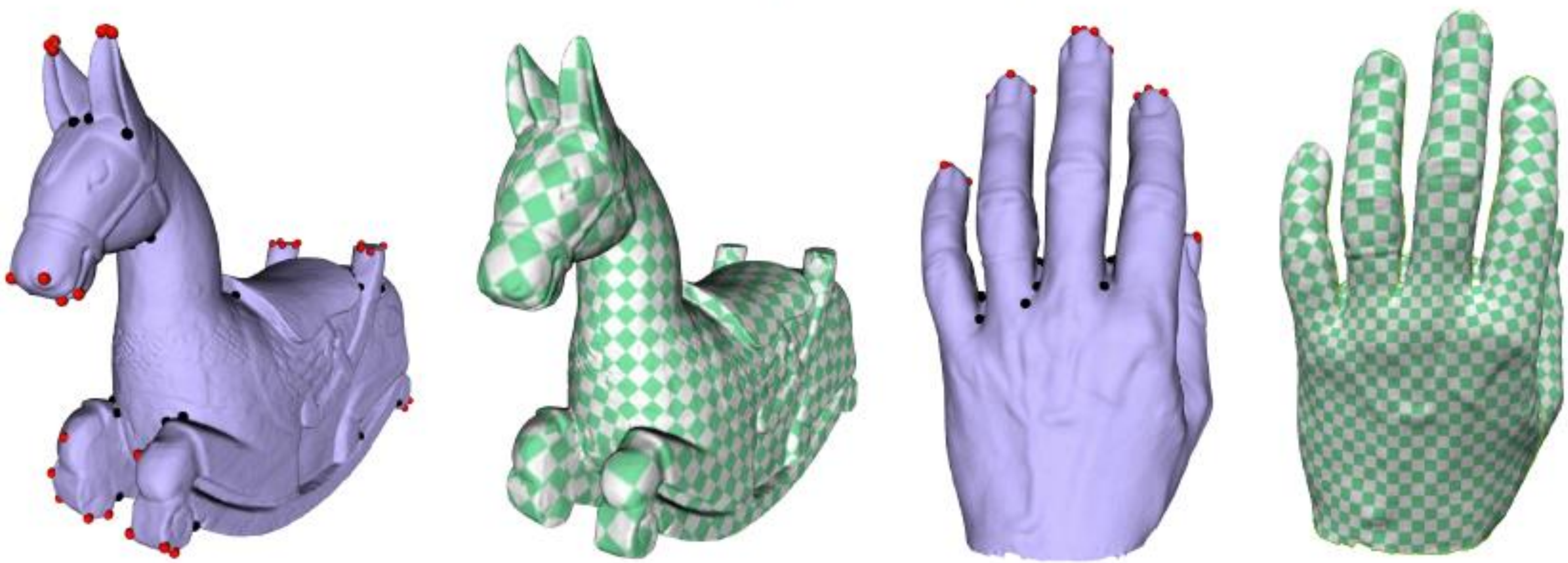
Parameterization

Sample Approach

Conformal Flattening by Curvature Prescription and Metric Scaling

Mirela Ben-Chen, Craig Gotsman and Guy Bunin

Technion – Israel Institute of Technology



Discretized Intrinsic Geometry

- **Discrete metric**
  Edge lengths
- **Discrete Gaussian curvature**
  Angle deficit from 2π;
  Feasible if it sums to $2\pi\chi(M)$

New Goal

Concentrate Gaussian curvature at a few singularities

Zero elsewhere!

Conformal Factor Relationship

Continuous

$$\Delta u = K^{orig} - e^{2u} K^{new}$$

Discrete

$$\Delta u = K^{orig} - K^{new}$$

Poisson equation for factor

Moving Curvature to Singularities

$$S = \text{Set of cones}$$

$$P_{ij} = \begin{cases} w_{ij}, & (i,j) \in E, i \notin S \\ 1, & i = j \text{ and } i \in S \\ 0, & \text{otherwise} \end{cases}$$

$$P^\infty = \begin{pmatrix} 0 & (I-S)^{-1}T \\ 0 & I \end{pmatrix}$$

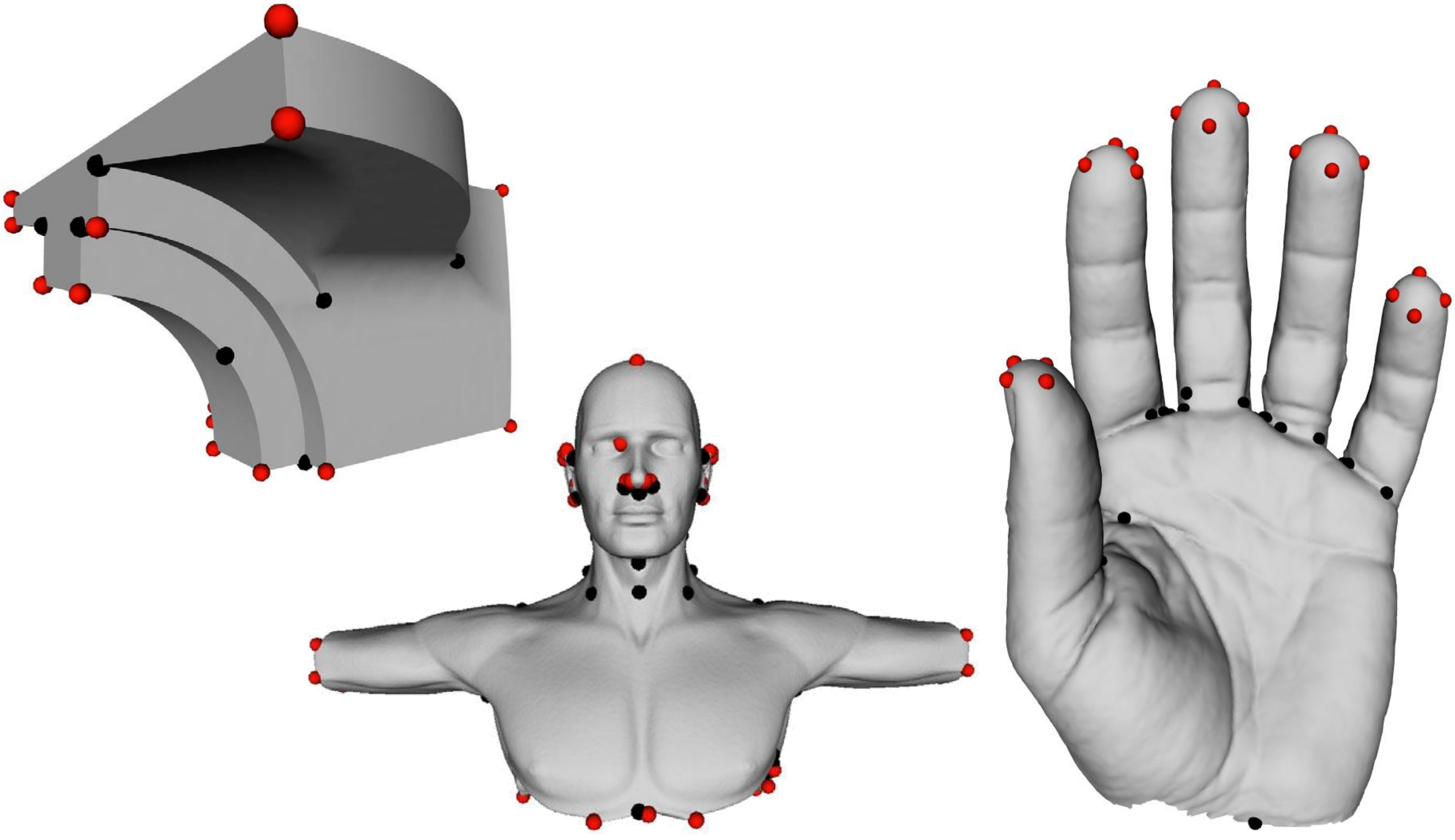
$$K_S^{new} = K_S^{orig} + ((I-S)^{-1}T)^\top K_{V \setminus S}^{orig}$$

Random walk with absorbing states

Placing Singularities

- **Initialize**
  All boundary vertices, plus extremal curvature point for nonzero characteristic
- **Move curvatures**
- **Compute conformal factor**
  Poisson equation
- **If large range, add points**
  Extrema of conformal factor; iterate

Singularity Examples



DDG Approach

Conformal Equivalence of Triangle Meshes

Boris Springborn
TU Berlin

Peter Schröder
Caltech

Ulrich Pinkall
TU Berlin

Abstract

We present a new algorithm for conformal mesh parameterization. It is based on a precise notion of *discrete conformal equivalence* for triangle meshes which mimics the notion of conformal equivalence for smooth surfaces. The problem of finding a flat mesh that is discretely conformally equivalent to a given mesh can be solved efficiently by minimizing a convex energy function, whose Hessian turns out to be the well known cot-Laplace operator. This method can also be used to map a surface mesh to a parameter domain which is flat except for isolated cone singularities, and we show how these can be placed automatically in order to reduce the distortion of the parameterization. We present the salient features of the theory and elaborate the algorithms with a number of examples.

Keywords: Discrete Differential Geometry; conformal parameterization; conformal equivalence; discrete Riemannian metric; cone singularities; texture mapping

Discrete Conformal Equivalence

- One conformal factor u_i per vertex

- Edge conformal equivalence:

$$\tilde{l}_{ij} = e^{(u_i+u_j)/2} l_{ij}$$

- Logarithmic lengths:

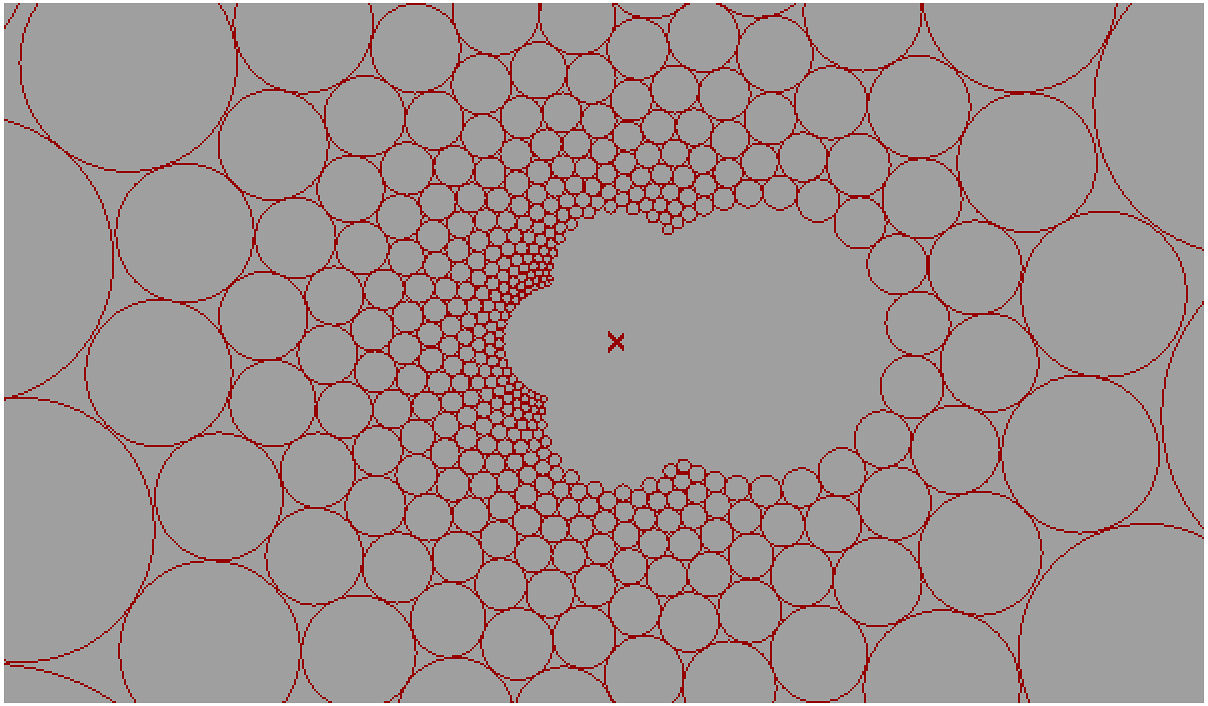
$$\tilde{\lambda}_{ij} = \lambda_{ij} + u_i + u_j$$

Condition for Conformality

Length cross ratios are preserved

$$c_{ij} \equiv \frac{\ell_{im}\ell_{jk}}{\ell_{mj}\ell_{ki}}, \text{ for triangles } t_{ijk}, t_{jim}$$

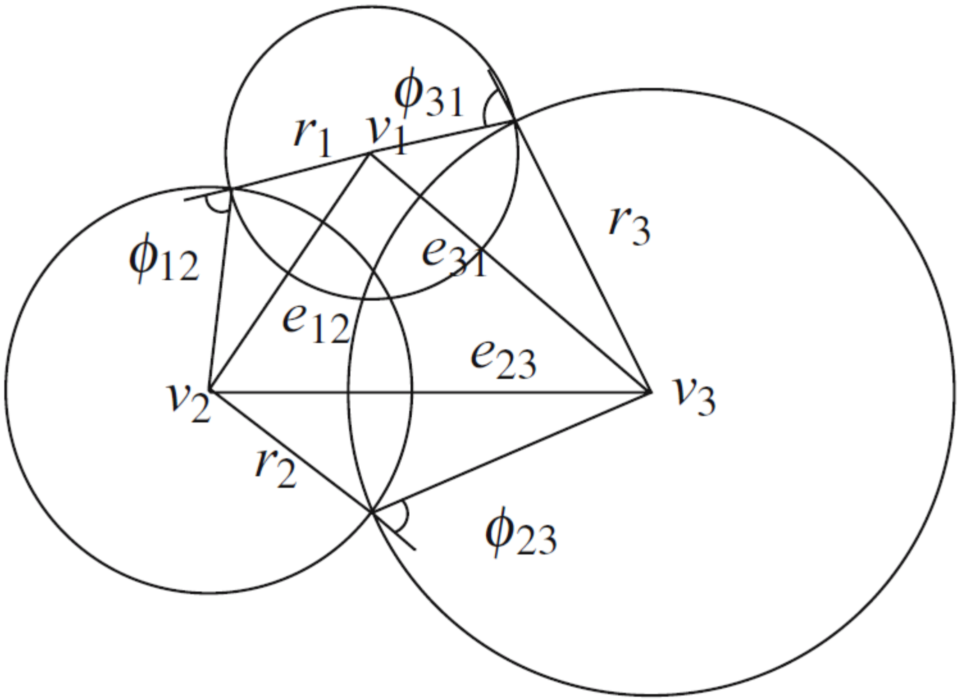
Third Idea for Conformality



http://www.designcoding.net/conformal-circle-packing/

Take circles to circles: "Circle packing"

Circle Packing



$$v_i = \text{vertex}$$

$$\gamma_i = \text{radius}$$

$$e_{ij} = \text{edge}$$

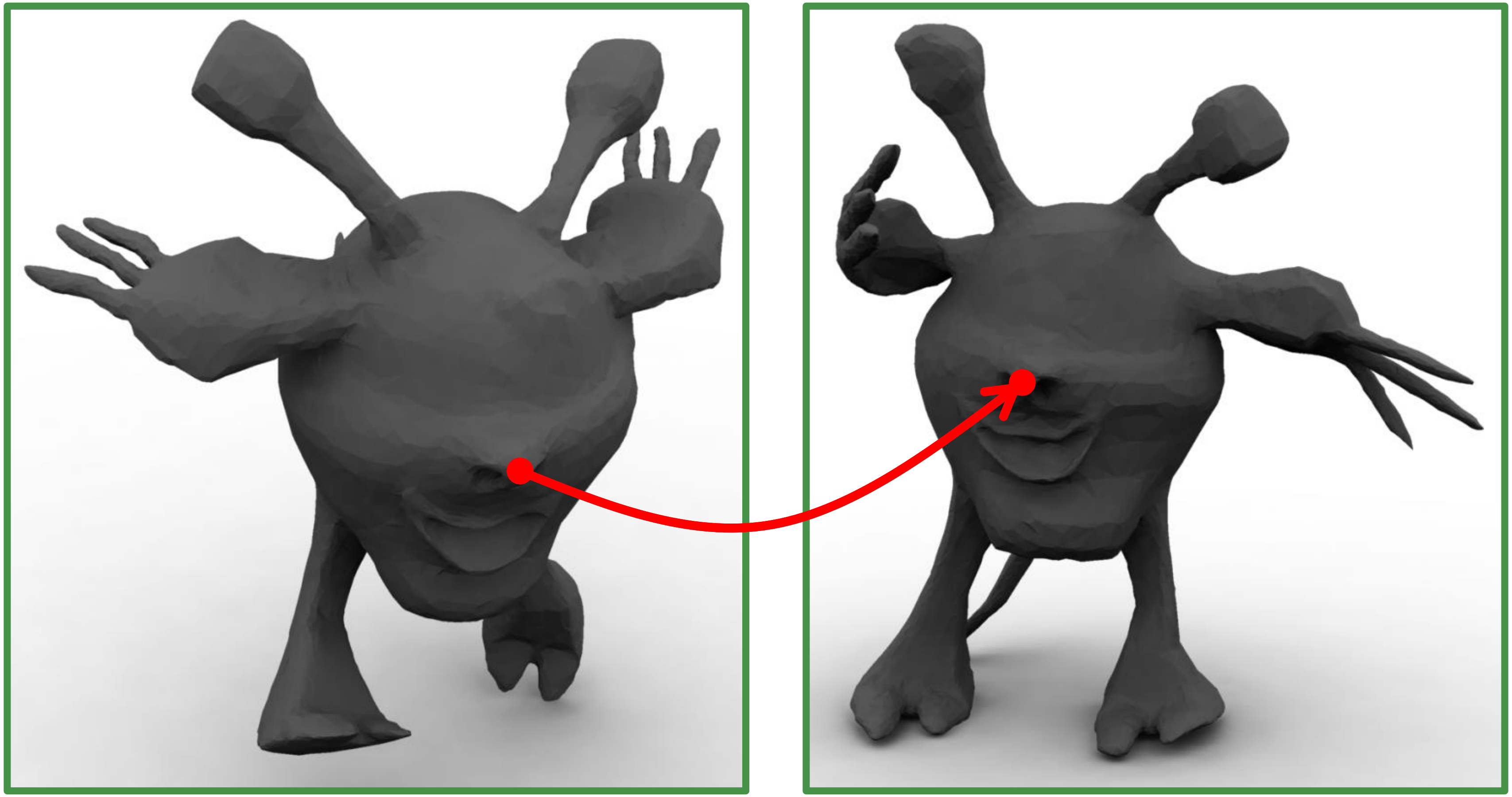
$$\Phi_{ij} = \text{circle intersection angle}$$

$$\ell_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

$$\text{Conformality: } \Phi_1 \equiv \Phi_2$$

http://link.springer.com/content/pdf/10.1007%2F978-3-540-68850-1_1.pdf

Second Application



Mapping

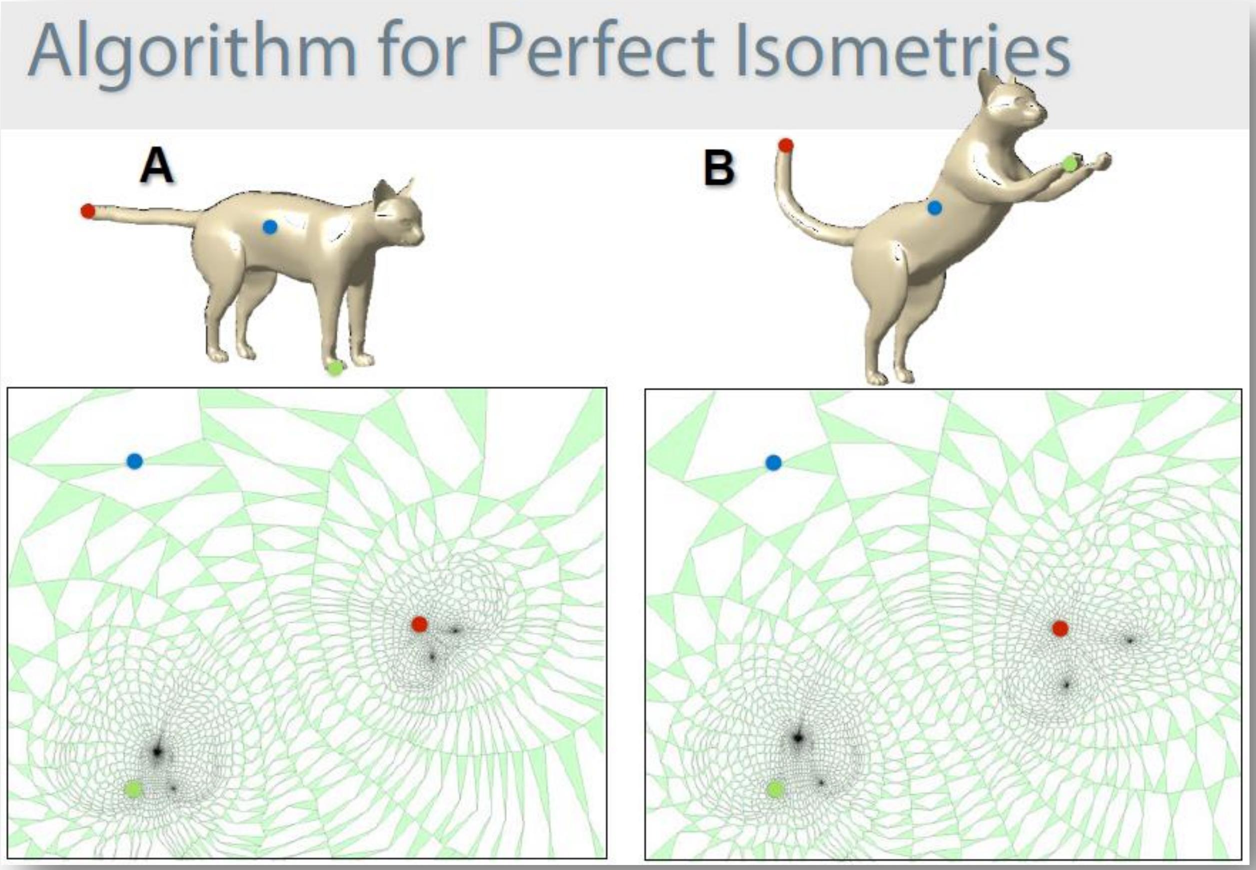
Observation About Mapping

$$\text{isometries} \subseteq \text{conformal maps}$$

Hard! Easier

Möbius Voting for Surface Correspondence
Lipman and Funkhouser 2009

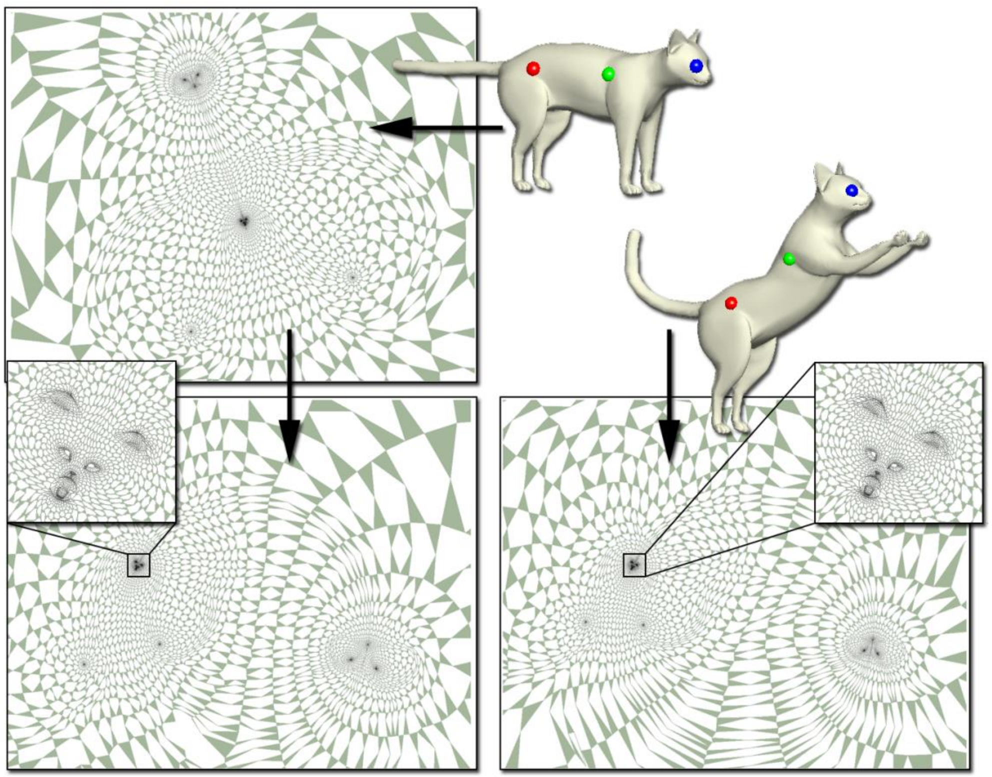
$O(n^3)$ Algorithm for Perfect Isometry



http://www.mpi-inf.mpg.de/resources/deformableShapeMatching/EG2011_Tutorial/slides/4.3%20SymmetryApplications.pdf

Map triplets of points

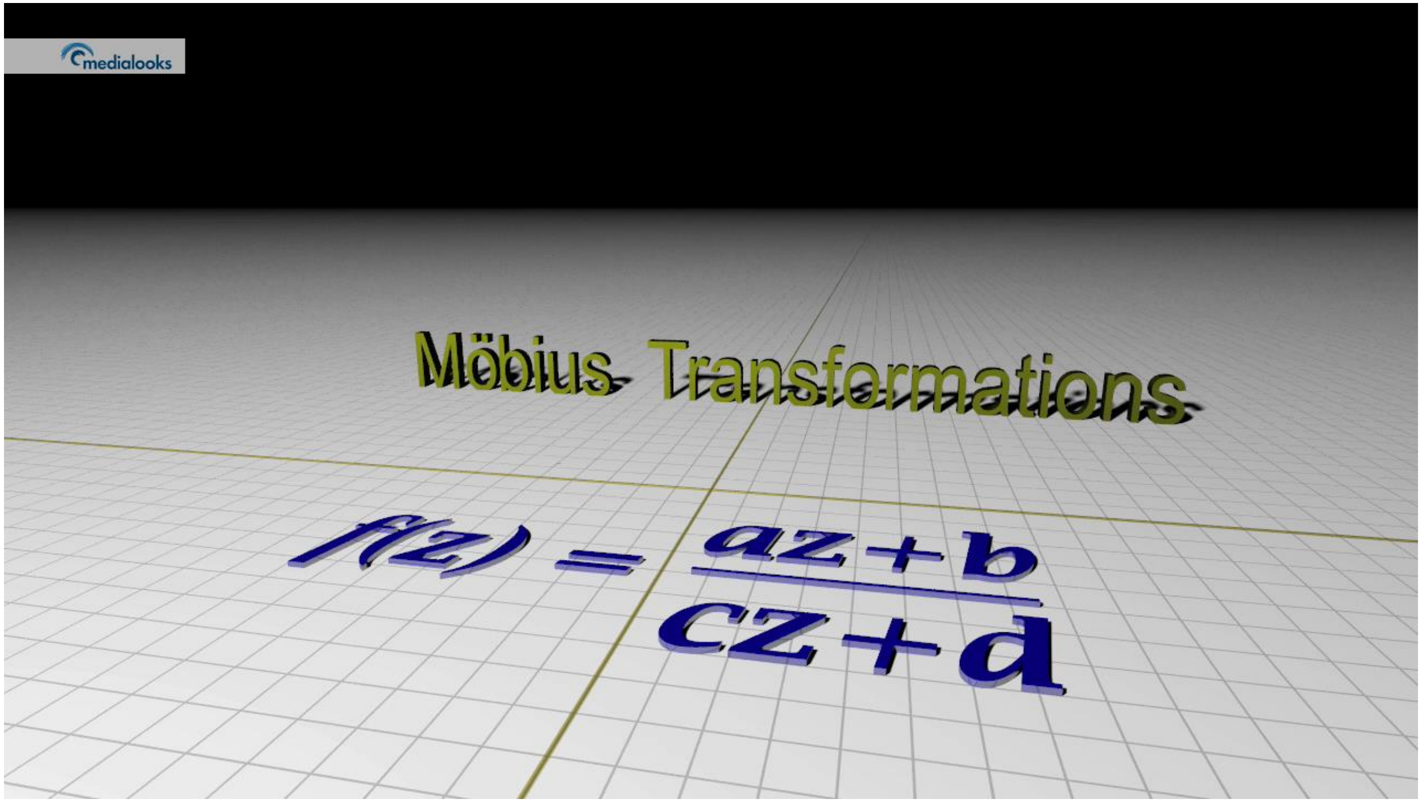
Möbius Voting



1. Map surfaces to **complex plane**
2. Select **three** points
3. **Map plane to itself** matching these points
4. **Vote** for pairings using distortion metric to weight
5. Return to 2

Möbius Voting for Surface Correspondence
Lipman and Funkhouser 2009

Möbius Transformations

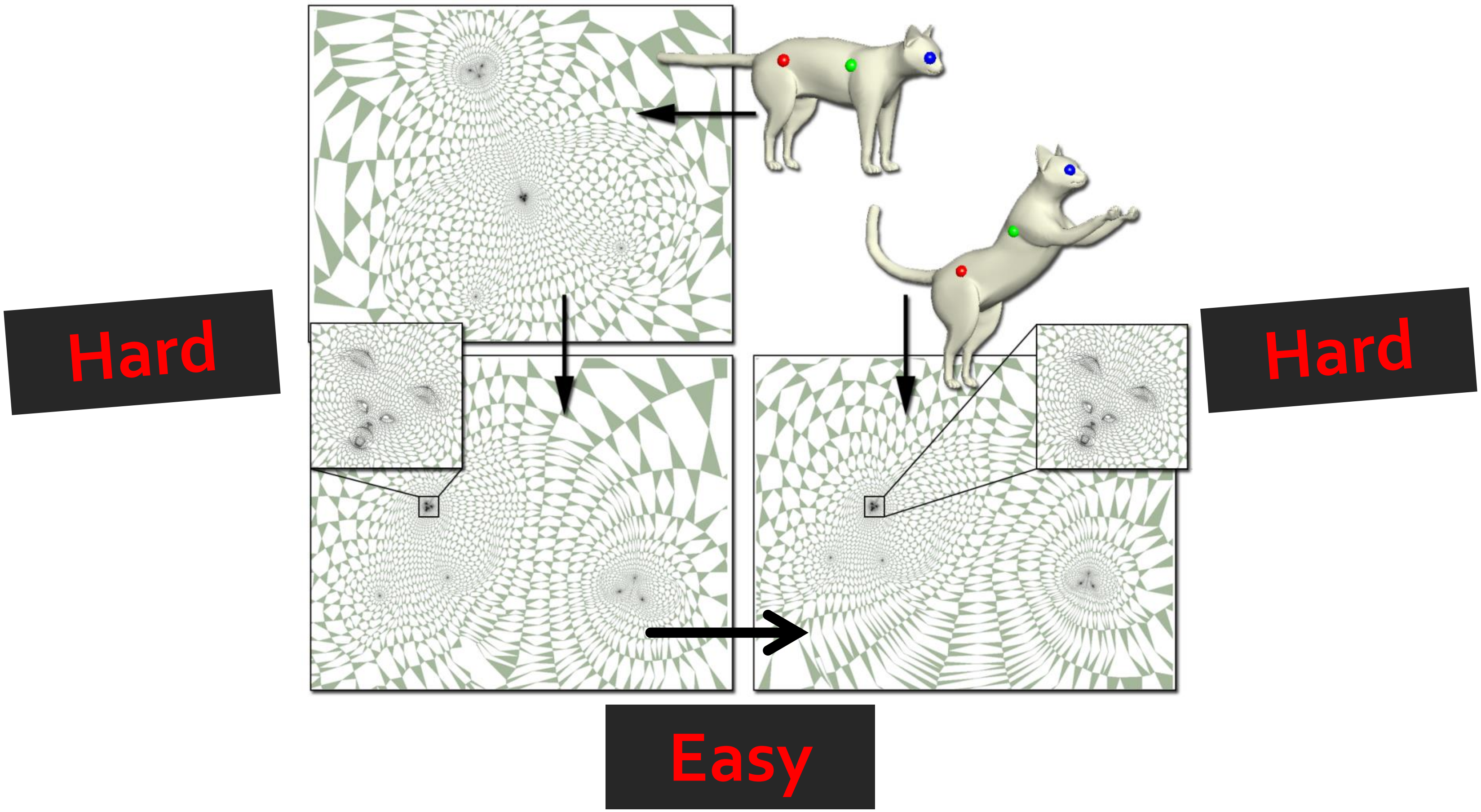


$$\frac{az+b}{cz+d}$$

http://www.ima.umn.edu/~arnold//moebius

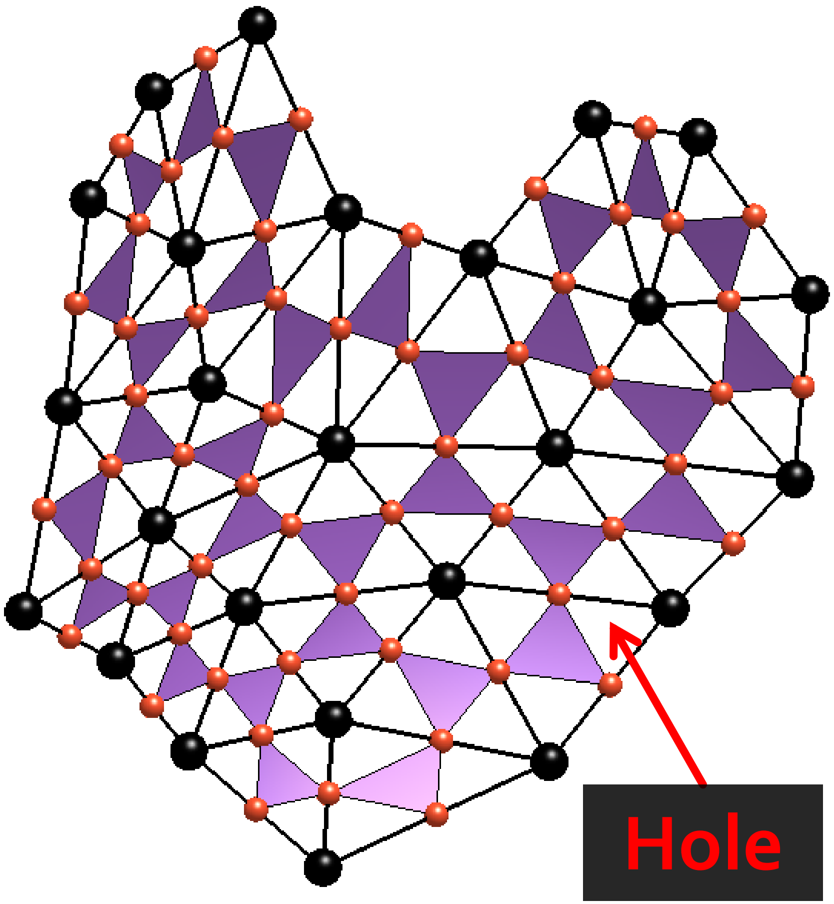
Bijective conformal maps of the extended complex plane

Observation



Hard work is per-surface, not per-map

Mid-Edge Flattening



$$\Phi(v) = u(v) + iu^*(v)$$

PL, continuous

$$\Delta u = 0$$

PL, continuous at midpoints

Rotate gradient of u 90°

Cannot scale triangles to flatten

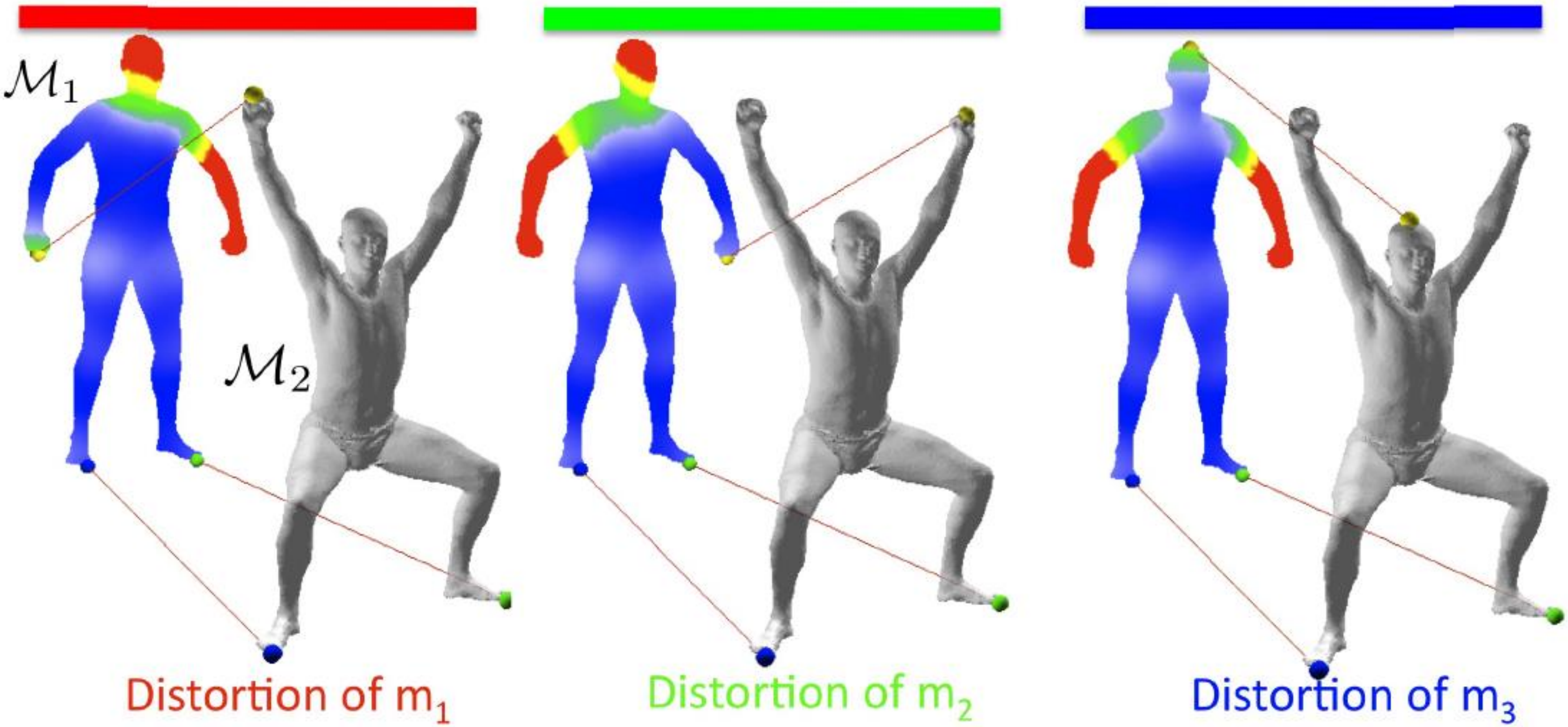
Voting Algorithm

Input: points $\Sigma_1 = \{z_k\}$ and $\Sigma_2 = \{w_\ell\}$
number of iterations I
minimal subset size K
Output: correspondence matrix $C = (C_{k,\ell})$.

```
/* Möbius voting */
```
while *number of iterations* $< I$ **do**
- Random $z_1, z_2, z_3 \in \Sigma_1$.
- Random $w_1, w_2, w_3 \in \Sigma_2$.
- Find the Möbius transformations m_1, m_2 s.t.
  $m_1(z_j) = y_j,\ m_2(w_j) = y_j,\ j = 1,2,3.$
- Apply m_1 on Σ_1 to get $\bar{z}_k = m_1(z_k)$.
- Apply m_2 on Σ_2 to get $\bar{w}_\ell = m_2(w_\ell)$.
- Find mutually nearest-neighbors $(\bar{z}_k, \bar{w}_\ell)$ to formulate candidate correspondence c.
- **if** *number of mutually closest pairs* $\geq K$ **then**
  - Calculate the deformation energy $\mathbf{E}(c)$
  - `/* Vote in correspondence matrix */`
  - **foreach** $(\bar{z}_k, \bar{w}_\ell)$ *mutually nearest-neighbors* **do**
    - $C_{k,\ell} \leftarrow C_{k,\ell} + \frac{1}{\varepsilon + \mathbf{E}(c)/n}$.
  - **end**
- **end**

end

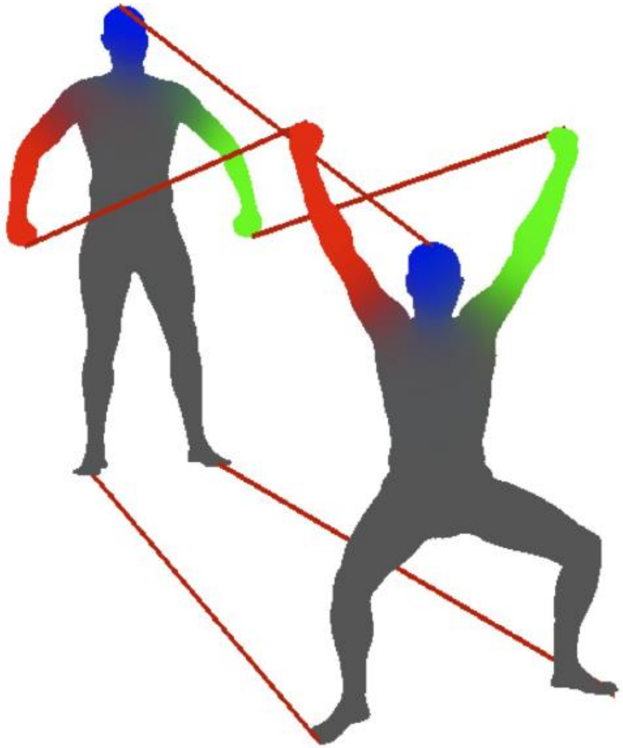
Use for Dense Mapping



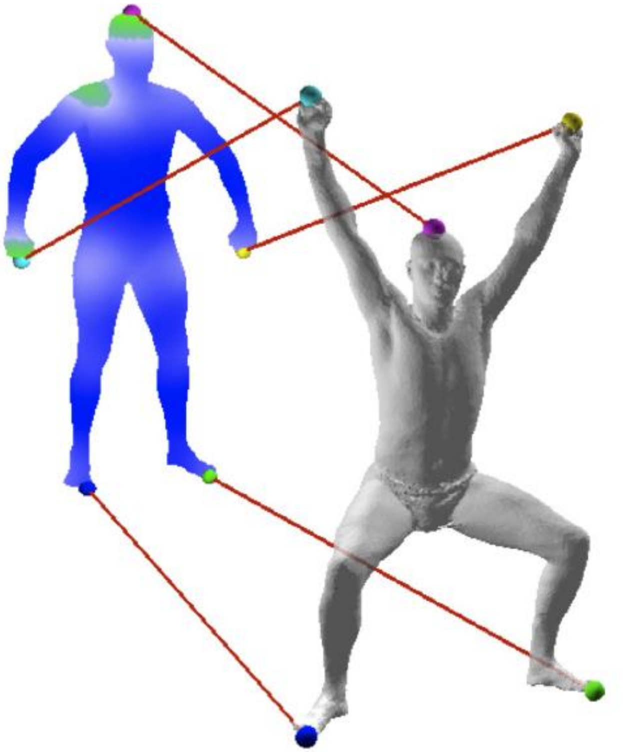
Different simple maps might be good in different places.

Blended Intrinsic Maps
Kim, Lipman, and Funkhouser 2011

Use for Dense Mapping



Blending Weights for m_1, m_2, and m_3

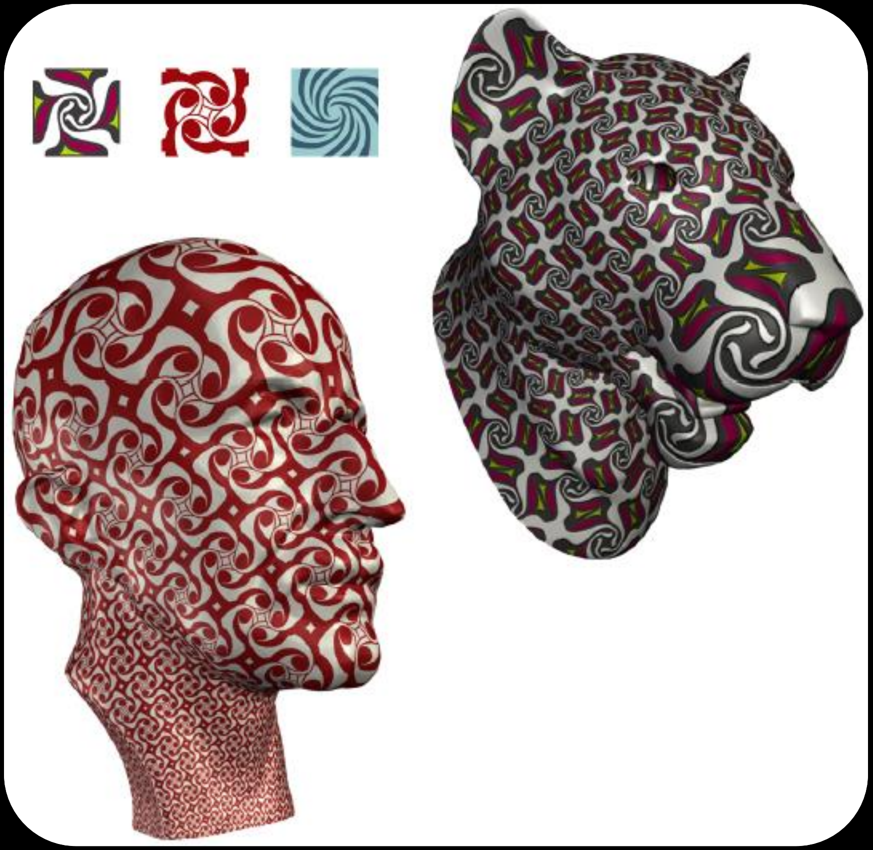


Distortion of the Blended Map

Combine good parts of different maps!

Blended Intrinsic Maps
Kim, Lipman, and Funkhouser 2011

http://www.multires.caltech.edu/pubs/ConfEquiv.pdf



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