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CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 19: Conformal Geometry

Conformal maps

In previous lectures we have explored the concept of isometries and we have seen that isometries are rare. However, there is a weaker condition: conformality. Whereas isometries preserve both lengths and angles, conformal maps preserve only angles.

Formally, let S_1, S_2 be surfaces with metrics g_1, g_2. A map $\phi : S_1 \to S_2$ is conformal if for all $X, Y \in T_pS_1$ $\exists$ function $u : S_1 \to \mathbb{R}$ s.t.:

$$g_2(D\phi_p(X), D\phi_p(Y)) = e^{2u(p)} g_1(X, Y) \tag{1}$$

Conformality is very flexible, in fact, all surfaces are locally conformal to the euclidean metric. This is stated in the following theorem:

Theorem: Let S be a surface. For every $p \in S$ there exists an isothermal parametrization for a neighbourhood of p. This parametrization satisfies the following: there exists $\mathcal{U} \subseteq \mathbb{R}^2$ and $\mathcal{V} \subseteq S$ containing p, a map $\phi : \mathcal{U} \to \mathcal{V}$ and a function $u : \mathcal{U} \to \mathbb{R}$ so that

$$g := [D\phi_x]^\top D\phi_x = \begin{pmatrix} e^{2u(x)} & 0 \\ 0 & e^{2u(x)} \end{pmatrix} \qquad \forall\, x \in \mathcal{U}$$

An important family of conformal maps comes from complex analysis. Swap $\mathbb{R}^2$ by $\mathbb{C}$, so now we have that for $p \in S$ the neighbourhood of p is holomorphic to a neighboorhood of $\mathbb{C}$.

The Uniformization theorem

The isothermal parametrization discussed in the previous section is local, however, one can ask what happens globally. This question leads to a very strong global statement:

Theorem: Let S be a 2D compact abstract surface with metric g. Then S possesses a metric $\bar{g}$ conformal to g with constant Gauss curvature $+1, -1$ or 0. Furthermore, S is conformal to a model space which is (the quotient by a finite group of self-conformal maps of) one of the following:

- The sphere with its standard metric if S has genus zero.
- The plane with its standard metric if S has genus one.
- The unit disk with the Poincaré metric if S has genus > 1.

A useful consequence of the uniformization theorem comes from the application of the Gauss-Bonnet formula. Under a conformal paramatrization, the Gaussian curvature transforms according to:

$$g_2 = e^{2u} g_1 \quad \Longrightarrow \quad K_2 = e^{2u}\big(-\Delta_1 u + K_1\big)$$

which, using the Gauss-Bonet formula leads to:

$$const. \times \mathsf{Area}(S) = \int_S K_2 \, dA_2 = 2\pi\chi(S)$$

that is, the sign of the uniformized curvature depends on topology.

For surfaces of genus zero, for example, we can find a parametrization conformal to the sphere. One key fact of such map is that it minimizes the Dirichlet energy $\mathcal{E}_D(\phi) := \int_S \|D\phi\|_{\mathsf{F}}^2 \, dA$.

For higher genus surfaces, however, there are more than one candidate for the conformal structure of S. We define the set $\mathcal{T}_S$ of conformal structures of S, called the Teichmüller space of S, whichis an abstract manifold of dimension

$$dim(\mathcal{T}_S) = \begin{cases} 2 & genus = 1 \\ 6g - 6 & genus = g > 1 \end{cases}$$

A parametrization of $\mathcal{T}_S$ is provided by holomorphic differentials:

- These are related to harmonic one-forms on S.
- The natural coordinates of $\mathcal{T}_S$ are the values of the line integrals of these differentials around homology generators of S.

Möbius transformations

These are conformal maps $\mathcal{M} : \mathbb{C} \cup \{\infty\}$ of especial interest, they are maps of the complex plane to itself given by:

$$f(z) = \frac{a\,z + b}{c\,z + d}$$

They can also be seen as conformal maps of the sphere $\mathbb{S}^2$ to the complex plane under stereographic projection, i.e., a Möbius transformation can be obtained by projecting the complex plane to the sphere, rotating and moving the sphere, and then projecting back to the complex plane.

Examples of conformal methods

Motivation

Application of conformal methods are numerous, among them, relevant examples are texture mapping, morphing and finding correspondences between two surfaces. To being able to create such maps requires especial considerations when we deal with discretized geometries such as those represented via triangular meshes. In the following sections we present some relevant examples.

Example 1: Conformal flattening by curvature prescription and metric scaling

A common procedure to find conformal parametrizations of triangular meshes is to cut the mesh and map each patch to a disk, however the main disadvantage of such approach is that it introduces discontinuities along the cuts and edges on either side of the cuts are scaled by different amounts. This is illustrated in figure 1.

Figure 1: Edge discontinuities common in mapping algorithms that cut the mesh first and then recompute the metric.

The example discussed here introduces a different approach to avoid those singularities [1]. The method is based on redistributing the secrete Gaussian curvature such that only a few vertices, termed *cone singularities*, concentrate all of the curvature and the rest have zero curvature. The cuts to the mesh are performed after the new metric is computed, therefore guaranteeing that edges on both sides of the cut are stretched by the same amount when mapped to the plane.

In addition to exploring the contributions of this manuscript, this brief overview of the paper allows for the review of basic discrete differential geometry tools and concepts.

Background

A triangular mesh $\mathcal{M}$ is given by a set of vertices V, edges E and faces F. The mesh is said to be embedded if there is an assignment from every vertex to a point in the physical space $X_{\mathcal{M}} := \{x_v \in \mathbb{R}^3 | v \in V\}$.

The discrete metric associated with this mesh is obtained by assigning a positive number to each edge on the mesh: $L_{\mathcal{M}} := \{l_{ij} \in \mathbb{R}^+ | (i,j) \in E\}$. The natural metric is the one in which the number assigned to each edge is its euclidean length.

The angles induced by a given metric at each vertex and for a given face are the result of applying the law of cosines to the edges' metric: $A_{L_{\mathcal{M}}} := \{\alpha_v^f = \arccos(l_{vu}^2 + l_{vw}^2 - l_{uw}^2)/(2l_{uv}l_{uw}\}$.

The discrete gaussian curvature induced by a given metric at a vertex is $K_{L_{\mathcal{M}}} := \{k_v = 2\pi - \sum \alpha_v^f | v \in V\}$.

Conformal map via curvature prescription

The goal is to find a conformal map that redistributes the Gaussian curvature to the singular vertices. Given the Gaussian curvature of a mesh, a target Gaussian curvature vector is called *feasible* as long as it satisfies the Gauss-Bonnet formula. Then, the problem is to find the new metric that induces the target Gaussian curvature and is conformal to the natural metric of the mesh.

From the continuous setting, under a conformal map, the Gaussian curvature changes according to:

$$\nabla^2 \phi = K^{\text{orig}} - e^{2\phi} K_{\text{new}}$$

In the discrete setting, however, the scaling factor disappears, and for each vertex we have the equation:

$$\nabla^2 \phi_v \approx k_v^{\text{new}} - k_v^{\text{orig}}$$

Where now the Laplacian can be replaced with the cotangent weights which is equivalent to the finite element approximation of the continuous Laplacian leading to the linear system:

$$\nabla^2 \phi = K^{\text{new}} - K^{\text{orig}}$$

The metric corresponding to such conformal map is a scaling of the natural metric of the original mesh. The scaling factors come from integration of the discrete map ϕ over the edge:

$$s_{ij} = \frac{e_j^{\phi} - e_i^{\phi}}{\phi_j - \phi_i} \; (i,j) \in E$$

Algorithm

First, a way to find a feasible Gaussian curvature vector is needed. The input is the original curvature vector and a set of singular vertices that will absorb all the curvature. We can think of the path going from the original vector of curvatures to the target vector as a Markov process. If a vertex is non-singular then it redistributes all of its curvature among its neighbors. If a vertex is singular, then it absorbs all the curvature from its neighbors. The closed form solution of the problem leads to:

$$K^{\text{new}} = K^{\text{orig}} + G\, K^{\text{orig}}$$

where each column of G is obtained solving a Laplacian system

$$L\, G_i = \delta_i$$

Finally, the algorithm is:

1. Initialize the set of cone singularities
2. Find the target curvatures
3. Compute the conformal map ϕ
4. Calculate the distortion $max(\phi) - min(\phi)$
5. It the distortion is greater than a user specified tolerance add $max(\phi)$ and $min(\phi)$ to the set of singularities and go back to 2.

Figure 2 illustrates the result of applying this algorithm.

Example 2: Conformal equivalence of triangle meshes

This manuscript follows a very similar path to the one discussed in Example 1, namely, the goal is to find a conformal parametrization such that the new parametrization has prescribed curvatures [3]. As in the previous example, the goal is to redistribute the Gaussian curvature so that it concentrates in some cone singularities and is zero everywhere else.

Figure 2: Results of the algorithm described in Example 1

Conformal equivalence

The concepts used here are the same as those employed in the previous section, namely, the notions of triangular mesh, discrete metric, defect angles and discrete Gaussian curvature. With those concepts at hand, the definition we seek comes from the question: What is conformal equivalence in the discrete setting? In the continuous case, two metrics are conformally equivalent if $\bar{g} = e^{2u}g$. For triangular meshes, two discrete metrics L and $\bar{L}$ are conformally equivalent if for some u_i associated with vertex $v_i \in V$ the two metrics are related by:

$$\bar{l}_{ij} = e^{(u_i+u_j)/2} l_{ij} \ |(i,j) \in E$$

This notion of discrete conformal equivalence can be regarded from the perspective of conserved quantities. We define the edge cross length ratio as:

$$\zeta_{ij} := l_{im}/l_{mj} \cdot l_{jk} l_{ki}$$

For an edge (i,j) and faces $f_{ink}, f_{jim} \in F$. Then, we have the proposition: Two meshes are conformally equivalent if and only if the length cross ratios are preserved.

Algorithm

The input is a mesh and a vector of target curvatures compatible with the Gauss-Bonnet formula just as in Example 1. The goal is to determine the corresponding conformal map based on the equivalence definitions provided in the previous section. The solution of such problem is that of solving the following convex minimization problem:

$$E(u) = \sum_{f_{ijk}} (f(\bar{\lambda}_{ij}, \bar{\lambda}_{jk}, \bar{\lambda}_{ki}) - \frac{pi}{2}(u_i + u_j + u_k)) + \frac{1}{2} \sum_{v_i} (\pi - k_i^{\text{new}}) u_i$$

whit:

$$f(\bar{\lambda}_{ij}, \bar{\lambda}_{jk}, \bar{\lambda}_{ki}) = \frac{1}{2} \left(al\bar{p}ha_i^{ijk} \bar{\lambda}_{jk} + al\bar{p}ha_j^{ijk} \bar{\lambda}_{ki} + al\bar{p}ha_k^{ijk} \bar{\lambda}_{ij} \right) + JI(al\bar{p}ha_i^{ijk}) + JI(al\bar{p}ha_j^{ijk}) + JI(al\bar{p}ha_k^{ijk})$$

where $\lambda_{ij} = 2\log l_{ij}$, $al\bar{p}ha_i^{ijk}$ is the angle at vertex i for face ijk and $JI(x) = -\int_0^x \log|2\sin t| \mathrm{dt}$. Figure 3 shows illustrative results from the application of this algorithm.

Example 3: Möbius voting for surface correspondence

Using the idea of mapping surfaces to the complex plane such that the resulting map is holomorphic opens the possibility of exploring the two flattened surfaces to search for correspondences [2]. The advantage of working in the complex plane is that we can find maps from the complex plane into itself via Möbius transformations which have closed form solution.

Figure 4 illustrates the problem to be solved, namely, given two surfaces, first map them conformally to the complex plane via Φ_1 and Φ_2, then relate the two complex planes using Möbius transformations.

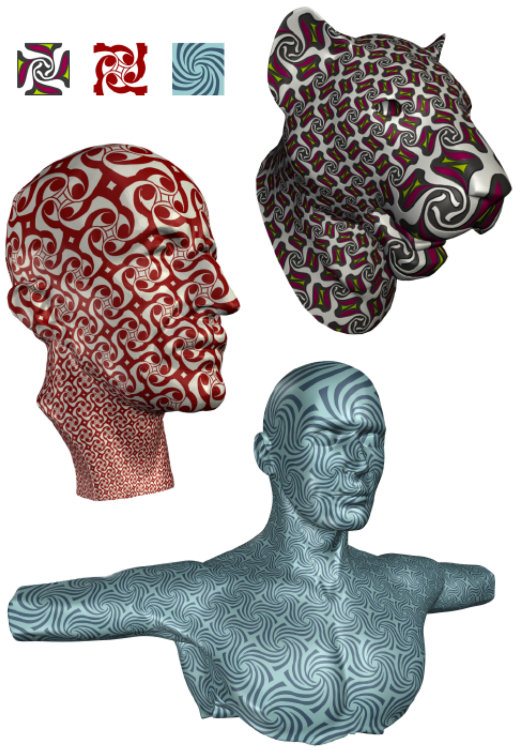

Figure 3: Results of the algorithm described in Example 2

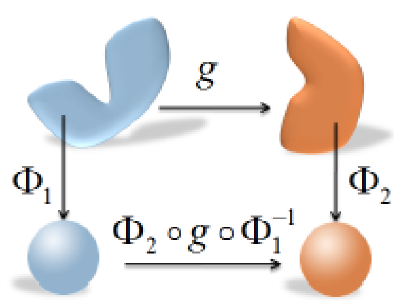

Figure 4: Schematic of the problem of finding correspondence points between surfaces

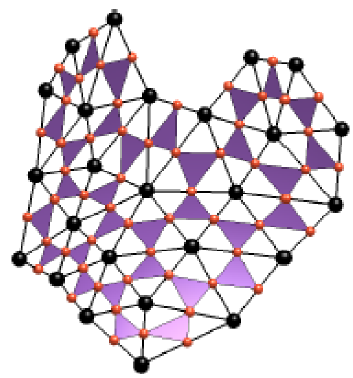

Figure 5: Mid-edge mesh

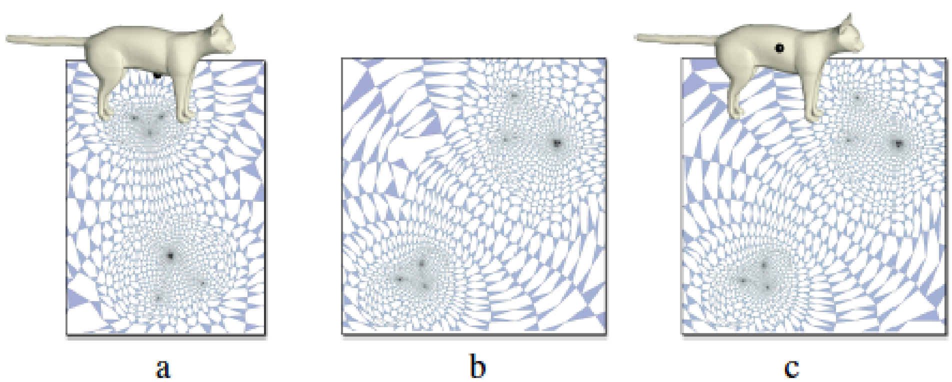

Figure 6: Example of the mid-edge uniformization

Mid-edge uniformization

There are several ways to achieve conformal maps. Examples 1 and 2 explore two novel techniques. The approach followed in this paper takes advantage of the mid-edge mesh shown in Figure 5. Black vertices are those from the original mesh, red vertices are the mid edges, and purple faces are the faces of the mid-edge mesh. Then, the goal is to find discrete conjugate harmonic functions $u(\cdot)$ and $u*(\cdot)$ to define the mapping $\Phi(v_i) = u(v_i) + \mathbf{i}u*(v_i)$. The discrete harmonic function can be obtained from:

$$\sum_{j \in \mathcal{N}} (u_i - u_j)(\cot \alpha_{ij} + \cot \beta_{ij}) = 0 \; \forall v_i \in V$$

where $\mathcal{N}$ is the one-ring of v_i and the angles α_{ij} and β_{ij} are the angles supporting the edge $(i, j) \in E$. Figure 6 illustrates such mapping. In principle, the map is not unique, but different maps are related by a single Möbius transformation. In the figure 6 , a) and b) are two different maps and c) is a transformation of b). Note the similarity between a) and c).

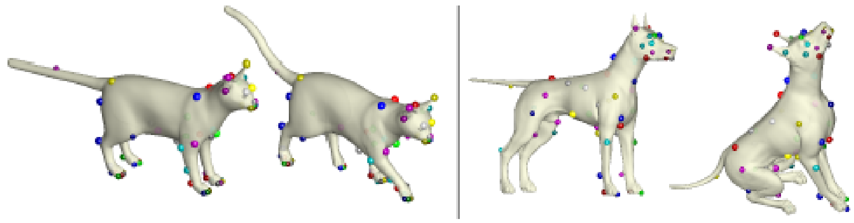

Figure 7: Results of the algorithm described in Example 3

Möbius voting algorithm

1. Select a subset of the vertices for each surface. These vertices are the ones for which we will test correspondence.

2. Perform Mid-edge uniformization to both surfaces

3. Select randomly three points from every subset and perform the corresponding Möbius transformation to map both complex parametrizations to a common canonical domain.

4. Evaluate the correspondence and fill in a fuzzy correspondence matrix.

5. Go to 3

   The advantage of this algorithm is that the hard step is the uniformization, but that is only done once for each surface. The iterations involve only Möbius transformations which are relatively simple. Figure 7 shows two examples.

References

[1] Mirela Ben-Chen, Craig Gotsman, and Guy Bunin. Conformal flattening by curvature prescription and metric scaling. In *Computer Graphics Forum*, volume 27, pages 449–458. Wiley Online Library, 2008.

[2] Yaron Lipman and Thomas Funkhouser. Möbius voting for surface correspondence. In *ACM Transactions on Graphics (TOG)*, volume 28, page 72. ACM, 2009.

[3] Boris Springborn, Peter Schröder, and Ulrich Pinkall. Conformal equivalence of triangle meshes. In *ACM Transactions on Graphics (TOG)*, volume 27, page 77. ACM, 2008.