CS 468

Differential Geometry for Computer Science

Lecture 17 — Surface Deformation

Outline

- Fundamental theorem of surface geometry.
- Some terminology: embeddings, isometries, deformations.
- Curvature flows
- Elastic deformations

The Gauss and Codazzi Equations

Recall the Gauss Equation:

$$
\begin{aligned}
0 &= \langle D_Y D_X Z - D_X D_Y Z, W \rangle \\
&= \mathrm{Rm}(X, Y, Z, W) + A(Y, Z)A(X, W) - A(X, Z)A(Y, W)
\end{aligned}
$$

The second important equation linking intrinsic and extrinsic geometry is the Codazzi Equation.

$$
\begin{aligned}
0 &= \langle D_Y D_X Z - D_X D_Y Z, N \rangle \\
&= \nabla A(X, Y, Z) - \nabla A(Y, X, Z)
\end{aligned}
$$

These are key consistency equations which in principle completely characterize the surface.

Fundamental Theorem of Surface Geometry

Theorem:

Let Ω be an open, simply-connected subset of the plane equipped with two tensor fields g and A satisfying the Gauss and Codazzi equations.

Then there exists a mapping $\phi : \Omega \to \mathbb{R}^3$ of class C^3 such that the first and second fundamental forms of the surface $M := \phi(\Omega)$ pull back to the tensor fields g and A.

ϕ is unique up to rigid motions.

Thus: The metric and second fundamental form determine the surface at least locally.

And: Changes to the surface can be characterized geometrically by how the metric and second fundamental form change.

Abstract Surfaces, Embeddings and Deformations

There is a notion of an abstract surface.

- This is a two-dimensional manifold that exists on its own, without reference to the ambient Euclidean space.

Let M be an abstract surface. A map $\phi : M \to \mathbb{R}^3$ is an embedding if it is a diffeomorphism onto its image and $\phi(x) = \phi(y)$ iff $x = y$.

- This is our "usual" definition of a surface.
- Let $S = \phi(M)$. Then M inherits a metric and a second fundamental form from S.
- Isometries are the changes of M that do not change the metric.
- Isometries of M may or may not involve changes of S.
  - $\rightarrow$ Rigid motions, a spherical cap, a developable surface.
- Deformations are changes of S that change both the metric and second fundamental form.

Curvature Flows

Controlled deformations of a surface arise in a number of ways.

E.g. a family ϕ_t of embeddings evolves by mean curvature flow if

$$\frac{d\phi_t}{dt} = H_t N_t$$

where H_t is the mean curvature of ϕ_t and N_t is their unit normal.

Note: We've seen this before. One can show that $\Delta\phi_t = H_t N_t$. This is Laplacian smoothing.

- So mean curvature is like heat flow except for surfaces! This wants to dissipate curvature.
- Analytical properties: short-time existence and smoothing.
- Non-linear — long-time existence in doubt, singularities

MCF of Curves in the Plane

A curve $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^2$ evolves by curve shortening flow if it satisfies

$$\frac{\partial \gamma_t}{\partial t} = k_t N_t$$

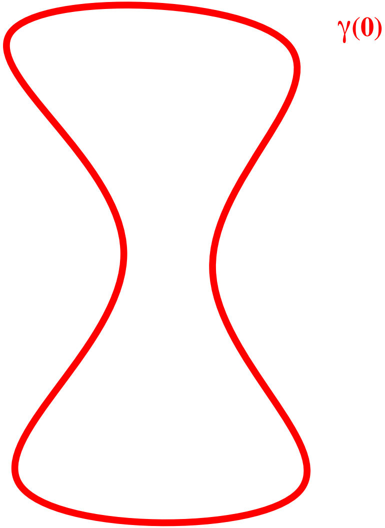
where k_t is the geodesic curvature of γ_t and N_t is its unit normal.

- Suppose that $\gamma_t = \gamma_t(s)$ is parametrized by arc length. By the Frenet formulas, the tangent vector satisfies $T_t := \frac{\partial \gamma_t}{\partial s}$ and

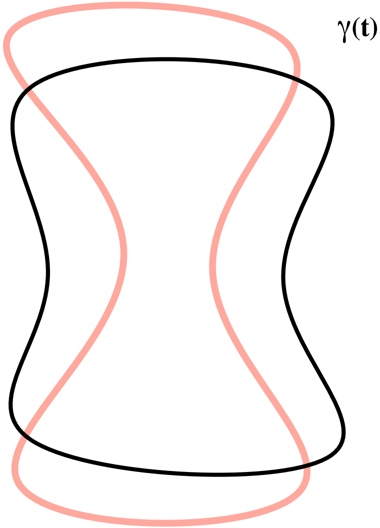
$$\frac{\partial^2 \gamma_t}{\partial s^2} = \frac{\partial T_t}{\partial s} = k_t N_t = \frac{\partial \gamma_t}{\partial t} \qquad \text{parabolic equations}$$

- Exact solution for a round circle — collapsing to a point.
- Some results.
  - → The Gage-Hamilton theorem for convex curves (preservation of convexity and convergence to a round point in finite time).
  - → The Grayson theorem for embedded curves (convergence to a round point in finite time).

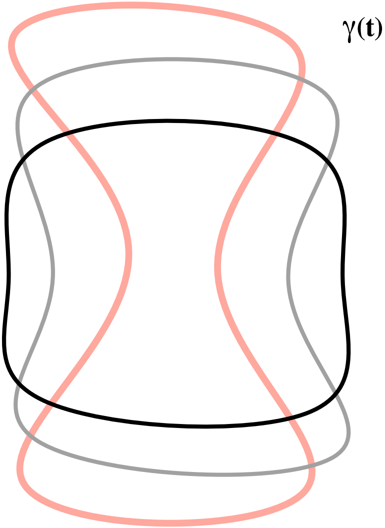
Grayson's Theorem



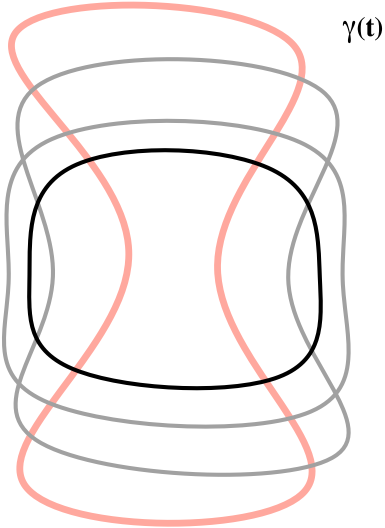
Grayson's Theorem



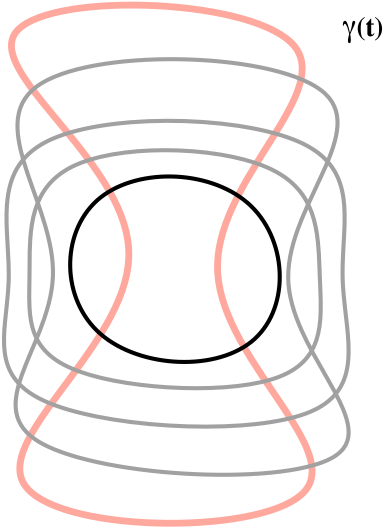
Grayson's Theorem



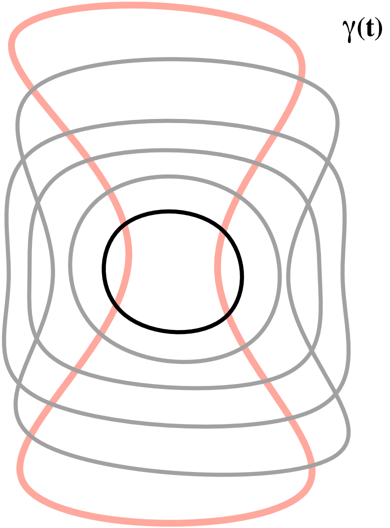
Grayson's Theorem



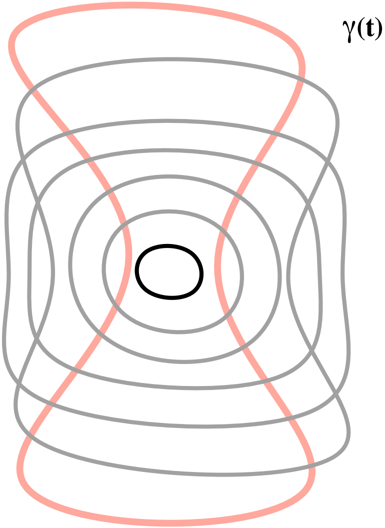
Grayson's Theorem



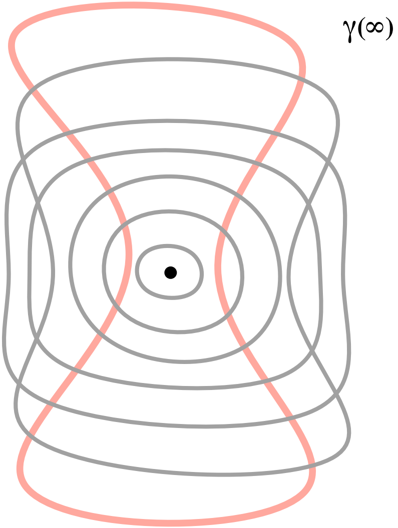
Grayson's Theorem



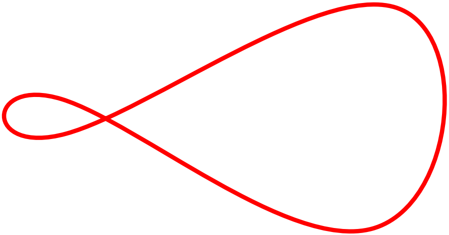
Grayson's Theorem



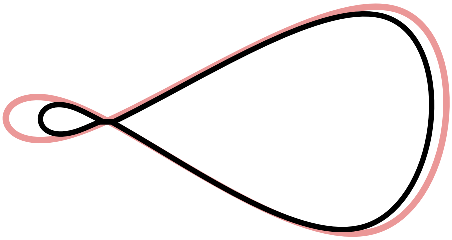
Grayson's Theorem



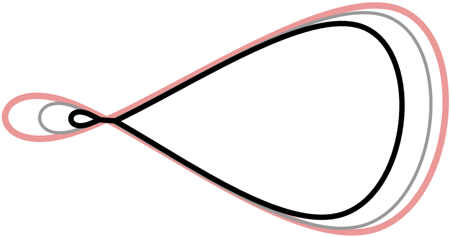
Singularity Formation in Grayson's Theorem



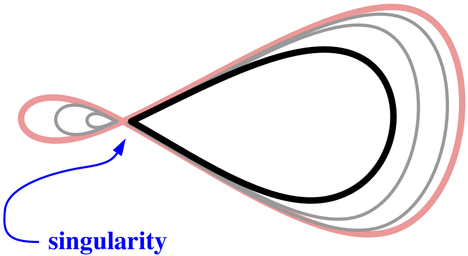
Singularity Formation in Grayson's Theorem



Singularity Formation in Grayson's Theorem



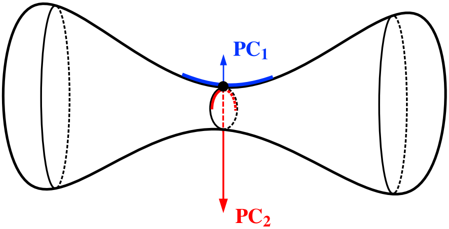
Singularity Formation in Grayson's Theorem



MCF of Surfaces

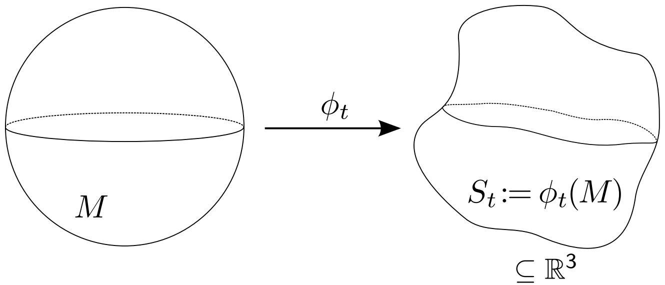
What changes for MCF of surfaces in $\mathbb{R}^3$?

- We still have a non-linear parabolic system.
- Exact solution for a round sphere — collapsing to a point.
- The Huisken theorem for convex surfaces (convergence to a round point in finite time).
- Singularities of the mean curvature flow in general — it's a tricky business! E.g. a dumb-bell surface.



Three-Dimensional Elasticity Theory

Elasticity theory characterizes deformations of an object by means of how they affect the induced metric in the reference object.



δ = Euclidean metric

$$g_{original} = \delta$$

$$g_{deformed} = \phi_t^* \delta := D\phi_t^\top D\phi_t$$

i.e. the pullback of δ under ϕ_t

Basic Principles

Let ρ be the density of M and $\rho_t := \rho \circ \phi_t^{-1}$ be the density of S_t. Also, let $v_t := \frac{\partial \phi_t}{\partial t} \circ \phi_t^{-1}$ be the spatial velocity of points in S_t.

The nonlinear equations of elasticity follow from three principles.

1. Mass is conserved:

$$\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v_t) = 0$$

2. Momentum is conserved:

Applied body forces $\downarrow$

$$\frac{\partial \rho_t v_t^i}{\partial t} + \sum_j \nabla_j (\rho_t v_t^j v_t^i) = \rho_t b_t^i + \sum_j \sigma_j^i N^j$$

$\uparrow$ Cauchy Stress Tensor
(The force per unit area on an internal surface element $\perp N$)

Basic Principles

The metric hasn't appeared yet. It encodes the **response** of the material to the applied forces.

Define the Dual Right Cauchy-Green Strain Tensor by

$$E = \frac{1}{2}\left(g_{deformed} - g_{orig}\right) = \frac{1}{2}\left(D\phi_t^\top D\phi_t - \delta\right)$$

Now we have our third principle.

3. The constitutive relation:

$$\sigma = \mathcal{P}(C \odot E)$$

where C is the elasticity tensor and $\mathcal{P}$ is the Piola transform that converts quantities in M to quantities in S_t.

Elastic Equilibrium

An object is in elastic equilibrium of ϕ_t is constant in t.

For hyperelastic materials we can characterize an equilibrium by means of a variational principle.

$$\phi_{equil} = \arg\min J(\phi) := \int_M W(x, E(x))dx$$

Here, W is the stored energy function. It can take many forms, depending on the material properties.

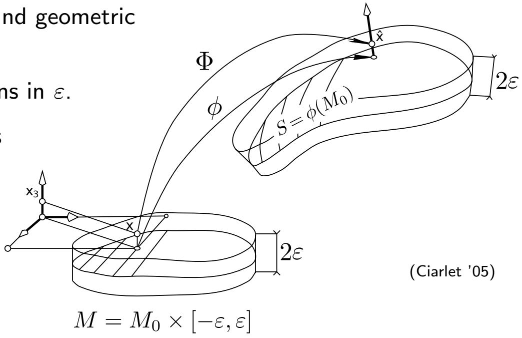
Elastic Shells

Consider a thin reference object $M := M_0 \times [-\varepsilon, \varepsilon]$ of thickness 2ε.

Propose the form $\Phi(x^1, x^2, x^3) := \phi(x^1, x^2) + x^3 N(x^1, x^2)$ for embedding M into $\mathbb{R}^3$, where $\phi : M_0 \to \mathbb{R}^3$ embeds M_0 as a surface.

The plan:

- Make several material and geometric hypotheses about Φ.
- Expand the 3D equations in ε.
- Derive formal equations satisfied by ϕ on S and M_0 alone.
- Prove convergence as $\varepsilon \to 0$.
- **Tricky business!**



(Ciarlet '05)

Elastic Equilibrium of Shells

Equilibrium configurations of shells can also be shown to minimize an energy functional.

$$\phi_{equil} = \arg\min_{\phi} \quad k_s \underbrace{\int_S C(\delta g, \delta g) dx}_{\text{stretching energy}} + k_b \underbrace{\int_S C(\delta A, \delta A) dx}_{\text{bending energy}}$$

where $\delta g := g_{orig} - g_{deformed}$ and $\delta A := A_{orig} - A_{deformed}$ and k_s, k_b are constants depending on assumptions and shell thickness.

Under certain assumptions on C, we can simplify to

$$\text{stretching energy} = \int_S \|g_{orig} - g_{deformed}\|^2 dx$$

$$\text{bending energy} = \int_S \|A_{orig} - A_{deformed}\|^2 dx$$