# CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 17: Surface Deformation

**Fundamental theorem of Riemannian geometry of surfaces.**

- Gauss-Codazzi equations.
  - Recall the Gauss equation from the proof of the Theorema Egregium.
  - Derivation of the Codazzi equation by working through $\nabla_X A(Y, Z) - \nabla_Y A(X, Z)$ for coordinate vector fields in $\mathbb{R}^3$.
- The fundamental theorem. Let $\Omega$ be an open, simply-connected subset of the plane equipped with two tensor fields $g$ and $A$ satisfying the Gauss-Codazzi equations. Then there exists a mapping $\phi : \Omega \to \mathbb{R}^3$ of class $C^3$ such that the first and second fundamental forms of the surface $M := \phi(\Omega)$ pull back to the tensor fields $g$ and $A$.
- Discussion of the proof.

**Curvature flows.**

- Deformations of the surface in space involve changes to the intrinsic and extrinsic geometry.
- Curve shortening flow in the plane.
  - Let $c_t : \mathbb{S}^1 \to \mathbb{R}^2$ with $t \in [0, T)$ be a family of solutions of the equation $\frac{\partial c_t}{\partial t} = k_{c_t} \vec{N}_t$ where $k_{c_t}$ is the geodesic curvature of $c_t$ and $\vec{N}_t$ is the normal vector of the curve. Then $c_t$ evolves by *curve shortening flow*.
  - Suppose that $c_t = c_t(s)$ is parametrized by arc length. By the Frenet formulas, the tangent vector of $c_t$ satisfies $\vec{X} = \frac{\partial c_t}{\partial s}$ and $\frac{\partial \vec{X}}{\partial s} = k_{c_t} \vec{N}$. Hence $\frac{\partial^2 c_t}{\partial s^2} = \frac{\partial c_t}{\partial t}$. This is a system of *parabolic equations*.
  - Exact solution for a round circle — collapsing to a point. Short time existence and finite-time singularity formation; but re-scaling leads to a nice solution for all time.
  - Preservation of convexity. The Gage-Hamilton theorem for convex curves (convergence to a round point in finite time).
  - The Grayson theorem for embedded curves (convergence to a round point in finite time).
- Mean curvature flow.
  - The equations are straightforward analogues of the curve shortening flow. Suppose $F_t : M \to \mathbb{R}^3$ is a family of immersions of a surface $M$ into $\mathbb{R}^3$ that satisfies $\frac{\partial F_t}{\partial t} = H_t \vec{N}_t$ where $H_t$ is the mean curvature of $F_t(M)$ and $\vec{N}_t$ is the unit normal vector of $F_t(M)$. Then $F_t(M)$ evolves by *mean curvature flow*.
  - This flow is such that the components of $F_t$ satisfy a system of non-linear, non-uniformly parabolic equations. But we have both short time existence and smoothing.
  - Easy to show $\frac{\partial \|\nabla u_t\|^2}{\partial t} < 0$ at $p_t := \arg\max \|\nabla u_t\|^2$ for solutions of the heat equation. Other intuitive smoothing arguments?
  - Exact solution for a round sphere — collapsing to a point. Short time existence and finite-time singularity formation; but re-scaling leads to a nice solution for all time.
  - The Huisken theorem for convex surfaces.
  - Singularities of the mean curvature flow in general — it's a delicate business! Example of what happens to a dumb-bell surface.

**Elasticity theory.** Let Justin introduce this material in Lecture 14. The point is not to exhaustively teach elasticity theory, but rather to give ten minutes of motivation for the bending and stretching energies of thin shells.

- Basic set-up of elasticity of domains.
- Basic set-up of elasticity of thin shells.
- Bending and stretching energies.