CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 14 | 15 May 2013 | Discrete Exterior Calculus

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1 Math Review

Before we dive into Discrete Exterior Calculus, let's review the topics from last lecture, with a focus on the methods that will be useful to us when operating on a triangle mesh. We will also try to develop more intuition behind tensors and the various operators discussed in the previous lecture.

1.1 Vector Calculus

The main objects we deal with in vector calculus are scalar functions and vector functions. A scalar function maps every point in space to a number, while a vector function maps every point in space to a vector. We have a few other essential objects and operators that will be integral to our study of exterior calculus:

Gradient: The gradient of a scalar function f is a vector field that represents the direction and magnitude of the greatest rate of increase of the function at each point:

$$\nabla f \equiv \sum_i \frac{\partial f}{\partial x_i}\hat{x}_i$$

Divergence: The divergence of a vector field $\vec{v}$ is a scalar function that represents a measure of how the vector field is spreading out at each point, or alternatively, a measure of the field's tendency to converge toward (sink) or repel from (source) a point:

$$\mathsf{div}\,\vec{v} \equiv \nabla \cdot \vec{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i}$$

Curl: The curl of a vector field $\vec{v}$ is a vector function that represents a measure of how the vector field is circulating at each point:

$$\mathsf{curl}\,\vec{v} \equiv \nabla \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

(shown in 3-space for simplicity)

Laplacian: Combining previous operators, we can take the divergence of the gradient of a function f to get the Laplacian:

$$\Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

The Laplacian of a f at a specific point p is roughly a measure of the rate at which the average value of f, over spheres centered at p, deviates from $f(p)$ as the sphere's radius increases (courtesy of Wikipedia). Not the most intuitive sentence, but the Laplacian is very useful in many practical applications and hopefully intuition will strike when we get there.

Let's also recall a couple of useful theorems from vector calculus (in $\mathbb{R}^2$):

Divergence Theorem: Suppose we have some domain Ω, a vector field with $\vec{v}$ defined at each point in Ω, and a normal vector $\vec{n}$ defined at each point along the boundary $\partial\Omega$ of our domain. The Divergence Theorem informally says that the total amount of "spreading-out" in the vector field is equal to the total amount that is leaving the boundary of the domain (see Fig. 1). More formally:

$$\int_{\Omega} \mathsf{div}\, \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{n}\, dl$$

Green's Theorem: Suppose we have some domain Ω, a vector field with $\vec{v}$ defined at each point in Ω, and a tangent vector $\vec{t}$ defined at each point along the boundary $\partial\Omega$ of our domain. Green's Theorem says something similar, but orthogonal, to the Divergence Theorem. Informally, it says that the total amount of circulation in the vector field is equal to the total amount that is circulating around the boundary of the domain (see Fig. 1). More formally

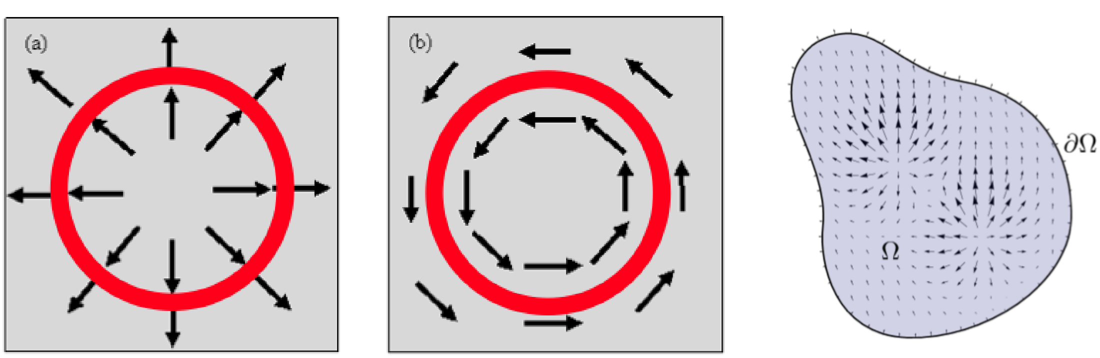
$$\int_{\Omega} \mathsf{curl}\, \vec{v}\, dA = \int_{\partial\Omega} \vec{v} \cdot \vec{t}\, dl$$

1.2 Exterior Calculus

We'd like to extend these various formalizations from vector calculus to surfaces (and manifolds). Up until now, we've been discussing abstract vector fields on some domain, with no constraints. Now, to consider our vector calculus machinery working on surfaces (and manifolds), we require the vector fields to be intrinsic, i.e. they are tangent to the surface at every point.

Differential Forms: One object that will be very useful to us from the study of tensors is something called a differential k-form. For each point p on a surface S, a differential k-form takes as input k vectors $v_1, \cdots, v_k$ where $v_i \in T_pS$, and outputs a real number. These k-forms have two important properties – they are multi-linear, or more specifically k-linear, and alternating or antisymmetric.

Example: Suppose we have a differential 2-form $\omega_p(v_1, v_2)$. The form is bilinear which means that $\omega_p(c_1v_1 + c_2v_2, v_3) = c_1\omega_p(v_1, v_3) + c_2\omega_p(v_2, v_3)$. The form is antisymmetric which means that $\omega(v_1, v_2) = -\omega(v_2, v_1)$.



(a) Divergence Theorem (b) Green's Theorem (c) Stokes' Theorem

Figure 1: Theorems visualized

Super Simple Example: A function f on a surface S is just a differential 0-form. It takes zero vector inputs, and outputs a real number, i.e. $f : S \to \mathbb{R}$. This is a trivial example, but still a very important one. We've learned many tools for differentiating functions on surfaces, and now the same tools will extend to differentiating forms and vector fields on surfaces.

One More Example: A differential 1-form takes as input a single vector x and outputs a real number. Suppose we have a surface S and some vector field $\vec{v}_p : S \to T_pS$, which assigns a vector to each point p on the surface. Associated with this vector field is a differential 1-form $\omega_p(x) = \langle v_p, x \rangle$ that takes a vector v at each point and outputs a real number, which is the inner product of the input vector against the vector field element at p. Finally, the association between these two objects $\vec{v}_p$ and ω_p is handled by the sharp and flat operators from last lecture:

$$(\vec{v}_p)^\flat = \omega_p \quad \text{and} \quad (\omega_p)^\sharp = \vec{v}_p$$

As a side-note, and a reminder of what the flat and sharp operators actually are, let's look at some intuition behind their nomenclature. Recall that upper and lower indices are important to us to differentiate between vector space indices (lower) and dual space indices (upper). In music, when we sharp a note we raise it by a half step; in differential geometry, when we sharp an element of a dual space we raise an index and get an element of the vector space! In music, when we flat a note we lower it by a half step; in differential geometry, when we flat an element of a vector space we lower an index and get an element of the dual space!

$$\omega^i = \sum_j g^{ij} v_j \quad \text{and} \quad v_i = \sum_j g_{ij} \omega^j$$

Integration of k-forms: We can integrate a k-form on a k-dimensional object – 1-forms on curves, 2-forms on surfaces, etc. Let's say we have a 1-form ω with an associated vector field $\vec{v}$ (i.e. $\omega^\sharp = \vec{v}$), and we want to integrate ω along a curve γ that lies on a surface S, where $\gamma : [0, 1] \to S$. Remember that we are concerned with vector fields that lie in the tangent space of our surface, so ω will be operating on the tangent vector γ':

$$\int_\gamma \omega = \int_0^1 \omega(\gamma')\, dt = \int_0^1 (\vec{v} \cdot \gamma')\, dt$$

What this integral tells us is the amount of our 1-form ω that is parallel to the curve γ. These integrals will be very useful when we start operating on a triangle mesh in discrete land.

Stokes' Theorem: One more piece of math review! Encapsulating nearly all these ideas and theorems we've seen so far, we have Stokes' Theorem. Suppose we have some domain Ω, and a form ω on that domain:

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

The intuition behind this theorem is very similar to the Divergence Theorem and Green's Theorem (see Fig. 1). One important note is that if Ω is a $(k+1)$-dimensional space, then ω should be a k-form. That way, on the LHS we are integrating a $(k+1)$-form $d\omega$ on a $(k+1)$-dimensional object Ω, and on the RHS we are integrating a k-form ω on a k-dimensional object $\partial\Omega$, the boundary of Ω.

2 Discrete Exterior Calculus (DEC)

We have a notion of discrete vs. discretized differential geometry. We consider a discrete version to be an application of a continuous concept to a discrete surface (a triangle mesh), where we make no approximations, but instead exactly model the continuous version. Whereas, a discretized version uses some approximations to reach a good enough estimate of some continuous concept. Here, we get to use the word discrete and not discretized (not entirely, as we'll see, but mostly). Let's first revisit a couple of useful concepts before continuing.

Oriented Simplicial Complex: A k-complex is a topological space constructed by essentially "gluing together" simplices (points, lines, triangles, etc). We're most interested in simplicial 2-complexes, a.k.a. triangle meshes. And more importantly, we care that they are oriented. What this means is we have a continuous normal field on the surface, but this isn't quite intuitive on a discrete surface. Instead we have to define a consistent direction for the normal vector at every point (inward vs. outward, or upward vs. downward, however you'd like to think of it), and another way to think of this is that every triangle must be specified in either a clockwise or counter-clockwise direction. Additionally, we'll be assigning an orientation to each edge, but this is for our benefit and not a requirement to consider the mesh oriented.

Dual Complex: If we have an n-complex, the dual complex is constructed by essentially taking each k-simplex and replacing it with it's $(n-k)$-simplex counterpart. For example, on a triangle mesh (n=2), each vertex (0-simplex) is transformed into a dual face (2-simplex), each edge (1-simplex) is transformed into a dual edge (1-simplex), and each face (2-simplex) is transformed into a dual vertex (0-simplex), and we get the dual mesh (see Fig. 2). We'll see later that this is where our discrete-ness becomes discretized, and we are forced to make some decisions and approximations in constructing the dual mesh. At this point, we just have topological rules for constructing the dual complex and, for example, no rule for where the dual vertices are located.

2.1 Differential Geometry on a Discrete Surface

Now we'd like to apply all our differential geometry techniques to a triangle mesh, but there's a problem here. How can we apply differential techniques that favor words like smooth, infinitesimal, etc, to a discrete surface that has a clear, fixed resolution (as long as we're not considering infinitely subdividing our mesh)?

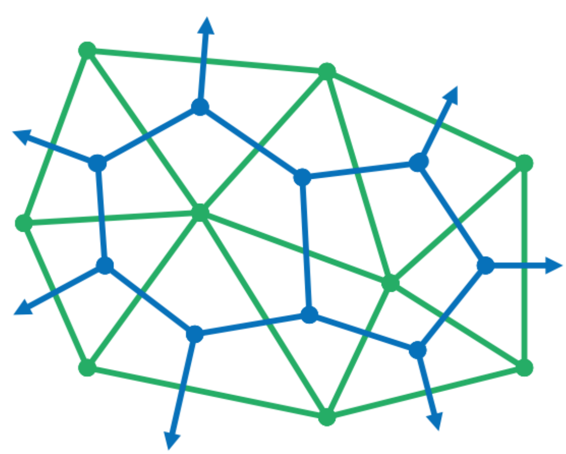


Figure 2: The dual complex (blue) on a triangle mesh (green).

The Trick: Recall from earlier that we have method for integrating these differential objects called k-forms on a k-dimensional object. This leads us to a very clean trick – instead of storing forms at discrete locations (vertices, edges, etc) on a mesh, we'll store integrals of those forms. Let's look at each case for a triangle mesh:

(1) *Discrete 0-form:* We'll store the integral of 0-forms on 0-simplices, i.e. the mesh vertices. A 0-form is just a function, and accordingly integrating over a vertex v simply gives the value of that function at the vertex:

$$\int_v \omega = f(v) \to \mathbb{R}^{|V|} \quad (V = \# \text{ of vertices})$$

(2) *Discrete 1-form:* Integrals of 1-forms will be stored on 1-simplices, i.e. the mesh edges. This is where choosing an edge orientation comes in to play. If we're integrating along an edge e in the opposite direction that it points, we can simply negate the integral:

$$\int_e \omega \to \mathbb{R}^{|E|} \quad (E = \# \text{ of edges})$$

(3) *Discrete 2-form:* Integrals of 2-forms will be stored on 2-simplices, i.e. the mesh faces. So, we'll store one value per face that represents the integral of our 2-form along that face:

$$\int_f \omega \to \mathbb{R}^{|F|} \quad (F = \# \text{ of faces})$$

2.2 Exterior Derivative

Now we can apply our new technique of integrating k-forms to calculate some of the differential quantities we've looked at in continuous land. Here, Stokes' Theorem is the key to the simplicity – the integral of the differential of a form on some domain is equal to the integral of the form along just the domain's boundary. We can apply this to calculate the exterior derivate of our k-forms (or at least, the integral of the exterior derivative, since we're storing integrated quantities).

0-form Example: Suppose we have a 0-form ω on our mesh and we want to determine its exterior derivative $d\omega$. Since ω is a 0-form, we know $d\omega$ is a 1-form, so we can integrate $d\omega$ along a 1-dimensional object, i.e. an edge in our mesh! Applying Stokes' Theorem gives us an integral over the boundary of the edge, and we know the boundary of an edge is simply its two vertices v_1 and v_2, where the edge points from v_1 to v_2:

$$\int_e d\omega = \int_{\partial e} \omega = \omega_{v_2} - \omega_{v_1}, \quad d \in \mathbb{R}^{|E|\times|V|}$$

1-form Example: If we want to determine the exterior derivative $d\omega$ of a 1-form ω on our mesh, the process is very similar. We know $d\omega$ is a 2-form and can therefore be integrated along a 2-dimensional object such as a face in our mesh. Stokes' Theorem again gives us an integral over the boundary of the triangle face, which is simply its three edges e_1, e_2, and e_3. One nuance is that edges have a direction that may or may not comply with the triangle face's orientation. To account for this, we simply negate the value ω_{e_i} stored in an edge e_i if it opposes the triangle's orientation.

$$\int_f d\omega = \int_{\partial f} \omega = \omega_{e_1} + \omega_{e_2} + \omega_{e_3}, \quad d \in \mathbb{R}^{|F|\times|E|}$$

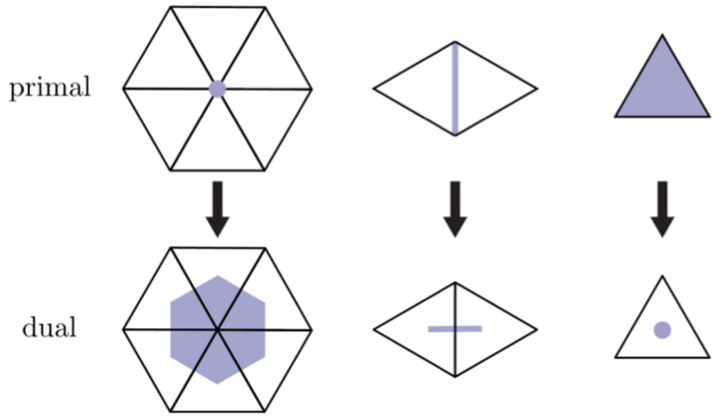


Figure 3: The Hodge star operator. *Left:* Star takes a 0-form on a vertex in the primal mesh to its corresponding 2-form on a face in the dual mesh. *Center:* Star takes a 1-form on an edge in the primal mesh to its corresponding 1-form on an edge in the dual mesh. *Right:* Star takes a 2-form on a triangle in the primal mesh to its corresponding 0-form on a vertex in the dual mesh.

2.3 Hodge Star

The idea behind the Hodge star operator is that it takes a k-form and returns a dual $(n-k)$-form. As we discussed earlier regarding the dual complex of a simplicial complex, this doesn't give us any indication of where these dual forms actually live spatially, and what value should be stored there. For example, if we apply the star operator on a 2-form stored on a triangle face, we get a 0-form back that should be stored on a vertex of the dual complex (see Fig. 3). Where should the dual vertex be located? This is where we'll introduce approximations, especially when going from a primal vertex to a dual face.

Primal 2-form to Dual 0-form: This is the simplest case of the star operator. When going from primal faces to dual vertices, the most sensible way to convert the integrated 2-form value over a face is to simply divide by the area of the face. Therefore, our Hodge star matrix to convert from 2-forms to 0-forms is nothing more than a diagonal matrix of one-over-area's:

$$\star_{ij} = \begin{cases} \frac{1}{\mathsf{Area}(\mathsf{triangle}_i)} & i = j \\ 0 & \text{otherwise} \end{cases}$$

Primal 1-form to Dual 1-form: Here we have to make some decisions. We're moving from edges to dual edges, so a natural method to convert the integrated 1-form over a primal edge e_i to its value on the dual edge $e_\star$ is to scale by the ratio of edge lengths:

$$\star\omega = \frac{|\ e_\star\ |}{|\ e_i\ |}\omega$$

Now, the problem is we haven't specified where the dual vertices exist spatially, and therefore have no idea where our dual edges live either. We only know topologically that there exists a dual edge between two dual vertices. We'll choose the dual vertices to be located at the circumcenter of their primal face counterparts (circumcenter is the center of the circle that passes through all three triangle vertices). This choice has some interesting implications if our mesh has obtuse triangles, but we'll disregard that for now. Still, doing this ensures that the dual edge $e_\star$ is perpendicular to e_i, and results in a nice (and familiar) formula for the edge length ratio in terms of the opposite angles α_i and β_i:

$$\frac{|\ e_\star\ |}{|\ e_i\ |} = \frac{1}{2}(\cot\alpha_i + \cot\beta_i)$$

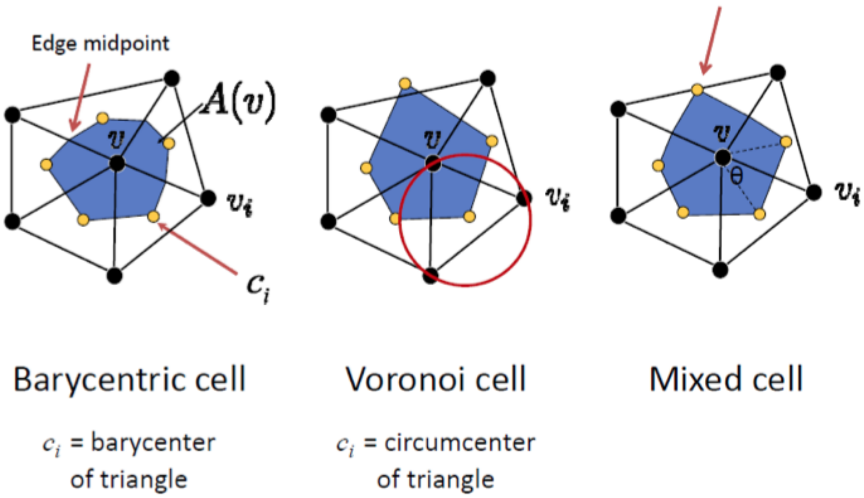


Figure 4: Choices of dual cell construction, partitioning area of a primal face. In the mixed Voronoi cell case, we place the dual vertices based on the Barycentric cell when the Voronoi cell would yield strange results (e.g. when a triangle is obtuse), and place the dual vertices based on the Voronoi cell otherwise.

Primal 0-form to Dual 2-form: Similar to the case of converting a primal 2-form to a dual 0-form, this will involve scaling by an area, specifically the area of the dual face. Again, we'll have a diagonal matrix of areas:

$$\star_{ij} = \begin{cases} \mathsf{Area}(\mathsf{cell}_i) & i = j \\ 0 & \text{otherwise} \end{cases}$$

Here we have another decision to make. How should the area of a primal face be partitioned amongst the dual faces? We could again go with placing dual vertices at the circumcenter of each primal face, then connecting the dots to construct essentially a Voronoi cell, but there are many options to choose from (see Fig. 4). In the case of the Barycentric cell, thin triangles can result in an odd dual cell that isn't necessarily convex. In the case of the Voronoi cell, we may have dual vertices that aren't actually contained within their primal face counterpart. The mixed cell method tries to take the best of both worlds by choosing one or the other based on the angle adjacent to the current primal vertex.

2.4 De Rham Complex

What we get out of all these constructions is something called a de Rham Complex (see Fig. 5). We have a nice way to go from primals to duals, primal 0-forms to 1-forms to 2-forms, and dual 2-forms to 1-forms to 0-forms. Additionally, we can go from primal 2-forms to 1-forms to 0-forms with one more operation called the co-differential δ, which can be expressed as simply a composition of other operations:

$$\delta \equiv - \star d \star$$

We can clearly see this equivalence in Fig. 5. Say we start with a primal 2-form on a primal face. Applying the star operator takes us to a dual 0-form on a dual vertex. Taking the differential gets us to a dual 1-form on a dual edge. And finally, another star operator brings us to a primal 1-form on a primal edge.

Note: Not every route through the de Rham Complex is meaningful or useful. In particular, a differential operator d_k operating on k-forms followed by another differential operator d_{k+1} operating on $(k+1)$-forms isn't useful, because it can be shown that the matrix $d_{k+1} d_k = 0$.

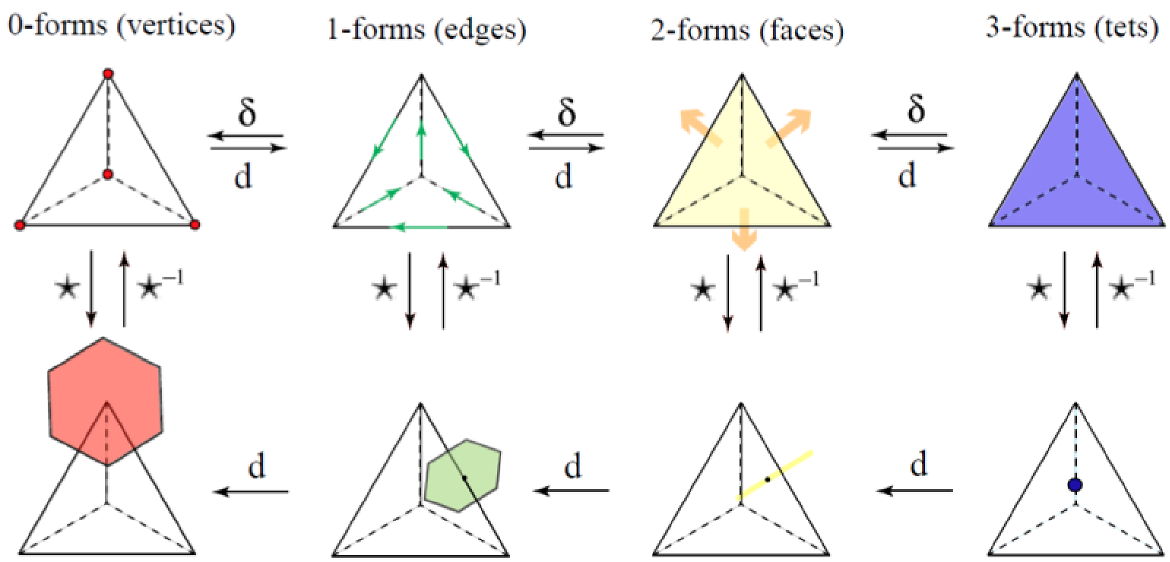


Figure 5: De Rham Complex. $\star =$ Hodge star operator, $d =$ differential operator, $\delta =$ co-differential operator $\equiv - \star d \star$.

2.5 Hodge Laplacian

We define another operation called the Hodge Laplacian as essentially all the ways you can move through the de Rham complex and get back to the form you started with:

$$\Delta = d \star d \star + \star d \star d$$

These are the only two ways of looping through and returning to the original k-form because, as the previous note stated, $d_{k+1} d_k = 0$, so essentially we can't move twice to the left in primal space or twice to the right in dual space (see Fig. 5).

0-form Example: Suppose we apply the Hodge Laplacian on a 0-form. Using the first term $d \star d \star$, if we take star of a 0-form we get a 2-form. Then taking the differential of a 2-form we get a 3-form. And a 3-form on a 2-dimensional object (our surface) is nothing (see last lecture for proof). So the first term is gone. Using the second term $\star d \star d$, it turns out that the $d \star d$ portion is nothing more than the cotangent Laplacian we've seen many times before, and the last $\star$ is just the area weights we defined earlier!

2.6 Helmholtz-Hodge Decomposition

Using these operators, we can decompose the vector field on a surface into three separate parts – a divergence-free or curl component, a curl-free or divergence component, and a harmonic component which is both divergence-free and curl-free (see Fig. 6).

Computing Decomposition: Suppose we have a 1-form ω (which has an associated vector field we're interested in decomposing), and we write it out as three components in the following way:

$$\omega = \delta\alpha + d\beta + \gamma$$

$$\delta\gamma = 0,\ d\gamma = 0$$

Now by applying the δ operator to the equation for ω, and recalling that applying d twice results in zero (and therefore applying δd does the same), we find an equation for $d\alpha$. Similarly, by applying

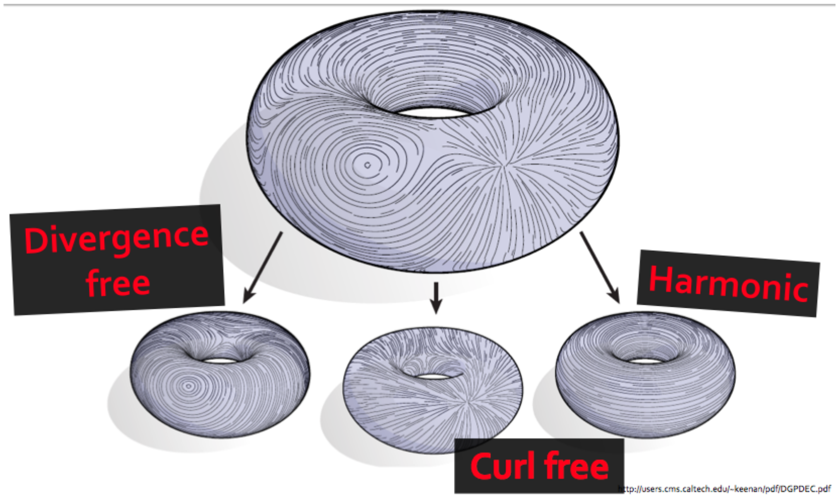


Figure 6: Helmholtz-Hodge Decomposition of a torus.

the d operator to the equation for ω, we find an equation for $\delta\beta$. Finally, we can find γ in terms of the others to get the following set of equations:

$$\delta d\alpha = \delta\omega$$

$$d\delta\beta = d\omega$$

$$\gamma = \omega - \delta\beta - d\alpha$$

3 Image Sources

- Figure 1 – http://brickisland.net/cs177/wp-content/uploads/2011/11/ddg_divergence_theorem.svg
- Figure 3 – http://brickisland.net/cs177/
- Figure 4 – http://graphics.stanford.edu/courses/cs468-12-spring/LectureSlides/05_Diff_Geo.pdf
- Figure 5 – http://ddg.cs.columbia.edu/SIGGRAPH06/DDGCourse2006.pdf
- Figure 6 – http://users.cms.caltech.edu/ keenan/pdf/DGPDEC.pdf