CS 468 (Spring 2013) — Discrete Differential Geometry

Lecture 13: Tensors and Exterior Calculus

Vectors, dual vectors and tensors.

- Let V be a real vector space with an inner product g.
- The dual space V^* and the sharp and flat opertors. Induced inner product.
- What happens when we choose a basis. The dual basis. Orthonormality.
- The tensor products $V^* \otimes \cdots \otimes V^*$. Induced inner product.
- Mixed tensors. The tensor product $V^* \otimes V$. Contraction.
- What happens when we choose a basis.

Symmetric and alternating tensors.

- Symmetrization and anti-symmetrization. Wedge product.
- What happens when we choose a basis. Dimension of the space of alternating tensors.
- The three non-trivial alternating tensor spaces over a two-dimensional vector space.
- Duality via the $*$-operator.

Tensor bundles on a surface.

- Definition of a linear bundle over a surface S.
- The tangent and cotangent bundles. Higher tensor bundles. Induced metric.
- Examples: the bundle of symmetric 2-tensors. and the bundle of k-forms.

Covariant differentiation in a linear bundle.

- The covariant derivative of vectors extends naturally to linear bundles.
- What happens when we choose a basis.

Exterior calculus.

- The exterior differential d.
- The co-differential $\delta := (-1)^{e(k)} * d *$ where k is the degree of the form acted upon.
- Formulas for divergence, gradient and curl.
- Relation to covariant derivatives.

Stokes Theorem.

- Chains. Boundary of a chain.
- Integration of a k-form over a k-chain.
- Stokes' formula $\int_c d\lambda = \int_{\partial c} \lambda$.

Topology and the de Rham complex.

- The de Rham complex.
- Laplacian on forms.
- Vector field decomposition.