A linear bound on the Complexity of the Delaunay Triangulation of Points on Polyhedral Surfaces

Dominique Attali
LIS, ENSIEG,
Domaine Universitaire, BP 46,
38402 Saint Martin d'Hères Cedex, France.

Jean-Daniel Boissonnat
INRIA,
2004 Route des Lucioles, BP 93,
06904 Sophia-Antipolis, France.

Abstract

Delaunay triangulations and Voronoi diagrams have found numerous applications in surface modeling, surface mesh generation, deformable surface modeling and surface reconstruction. Many algorithms in these applications begin by constructing the three-dimensional Delaunay triangulation of a finite set of points scattered over a surface. Their running-time therefore depends on the complexity of the Delaunay triangulation of such point sets.

Although the Delaunay triangulation of points in $\mathbb{R}^3$ can be quadratic in the worst-case, we show that, under some mild sampling condition, the complexity of the 3D Delaunay triangulation of points distributed on a fixed number of facets of $\mathbb{R}^3$ (e.g. the facets of a polyhedron) is linear. Our bound is deterministic and the constants are explicitly given.

Keywords: Delaunay triangulation, Voronoi diagram, complexity, polyhedral surfaces, sample, surface modeling, reconstruction.

1 Introduction

Delaunay triangulations and Voronoi diagrams are among the most thoroughly studied geometric data structures in computational geometry. Recently, they have found many applications in surface modeling, surface mesh generation [12], deformable surface modeling [20, 15], medial axis approximation [4, 9, 19], and surface reconstruction [3, 1, 8, 2, 7, 6]. Many algorithms in these applications begin by constructing the three-dimensional Delaunay triangulation of a finite set of points scattered over a surface. Their running-time therefore depends on the complexity of the Delaunay triangulation of such point sets.

It is well known that the complexity of the Delaunay triangulation of n points in $\mathbb{R}^d$, i.e. the number of its simplices, can be $\Omega(n^{\lceil \frac{d}{2} \rceil})$ [10]. In particular, in $\mathbb{R}^3$, the number of tetrahedra can be quadratic. This is prohibitive for applications where the number of points is in the millions, which is routine nowadays. Although it has been conjectured for a while that the complexity of the Delaunay triangulation of well-sampled surfaces should be linear, no result close to this bound has been obtained yet. Our goal is to exhibit practical geometric constraints that imply subquadratic and ultimately linear Delaunay triangulations. Since output-sensitive algorithms are known for computing Delaunay triangulations [11], better bounds on the complexity of the Delaunay triangulation would immediately imply improved bounds on the time complexity of computing the Delaunay triangulation.

First results on Delaunay triangulations with low complexity have been obtained by Dwyer [13, 14] who proved that, if the points are uniformly distributed in a ball, the *expected* complexity of the Delaunay triangulation is only linear. Recently, Erickson [16, 17] investigated the complexity of three-dimensional Delaunay triangulations in terms of a geometric parameter called the spread, which is the ratio between the largest and the smallest interpoint distances. He proved that the complexity of the Delaunay triangulation of any set of n points in $\mathbb{R}^3$ with spread Δ is $O(\Delta^3)$.

Despite its practical importance, the case of points distributed on a surface has not received much attention. A first result has been obtained by Golin and Na [18]. They proved that the expected complexity of 3D Delaunay triangulations of random points on any fixed *convex* polytope is $\Theta(n)$. Recently, Attali and Boissonnat proved that, for any fixed polyhedral surface S, any so-called "light-uniform ε-sample" of S has only $O(n^{7/4})$ Delaunay tetrahedra, or $O(n^{3/2})$ if the surface is convex, where n is the number of sample points. Applied to a fixed C^2 uniformly-sampled surface, the result of Erickson mentionned above shows that the Delaunay triangulation has complexity $O(n^{3/2})$. This bound is tight in the worst-case. It should be noticed however that Erickson's definition of a uniform sample is rather restrictive and does not allow two points to be arbitrarily close (in which case, the spread would become infinite).

In this paper, we consider the case of points distributed on a fixed number of planar facets in $\mathbb{R}^3$, e.g. the facets of a given polyhedron. Under a mild uniform sampling condition, we show that the complexity of the Delaunay triangulation of the points is linear. Our bound is deterministic and the constants are explicitly given.

2 Notations and definitions

2.1 Notations

For a curve Γ, we denote by $l(\Gamma)$ its length. For a portion of a surface R, we denote by $a(R)$ its area, and by ∂R its boundary. We further denote by $B(x, r)$ ($\Sigma(x, r)$) the ball (sphere) of radius r centered at x, and by $D_p(x, r)$ the disk lying in plane P centered at $x \in P$ and of radius r.

Let $R \subset P$ be a region of P. The plane P containing R is called a *supporting plane* of R. We define:

$$R \oplus_p \varepsilon = \{x \in P : D_p(x, \varepsilon) \cap R \neq \emptyset\}$$
$$R \ominus_p \varepsilon = \{x \in P : D_p(x, \varepsilon) \subset R\}$$

$R \oplus_p \varepsilon$ is obtained by growing R by ε within its supporting plane P and $R \ominus_p \varepsilon$ is obtained by shrinking R by ε within its supporting plane P. When the supporting plane is unique or when it is clear from the context, we will simply note $R \oplus \varepsilon$ and $R \ominus \varepsilon$.

2.2 Polyhedral surfaces

We call *polyhedral surface* a finite collection of bounded polygons, any two of which are either disjoint or meet in a common edge or vertex. The polygons are called *facets*. Notice that we allow an arbitrary number of polygons to be glued along a common edge. In the mathematical literature, such an object is called a pure two-dimensional piece-wise linear complex. We prefer to use in this paper the term surface since surfaces are our primary concern.

In the rest of the paper, S denotes an arbitrary but fixed polyhedral surface. Three quantities C, A and L will express the complexity of the surface S : C is the number of facets of S, $A = a(S)$ its area, and L the sum of the lengths of the boundaries of the facets of S:

$$L = \sum_{F \subset S} l(\partial F).$$

Observe that, if an edge is incident to k facets, its length will be counted k times.

We consider two zones on the surface, the ε-singular zone that surrounds the edges of S and the ε-regular zone obtained by shrinking the facets.

Definition 1 (ε-regular and ε-singular zones) *Let $\varepsilon \geq 0$. The ε-regular zone of a facet $F \subset S$ consists of the points of F at distance greater than ε from the boundary of F. The ε-regular zone of S is the union of the ε-regular zones of its facets. We call ε-*singular zone *of F (*resp. *S) the set of points that do not belong to the ε-regular zone of F (*resp. *S).*

Observe that the ε-regular zone of the facet F is $F \ominus \varepsilon$. The 0-singular zone of S consists exactly of the edges of S.

2.3 Sample

Any finite subset of points $E \subset S$ is called a *sample* of S. The points of E are called *sample points*. We impose two conditions on samples. First, the facets of the surface must be *uniformly* sampled. Second, the sample cannot become arbitrarily dense locally.

Definition 2 ((ε, κ)-sample) *Let S be a polyhedral surface. $E \subset S$ is said to be a (ε, κ)-sample of S iff for every facet F of S and every point $x \in F$:*

- *the ball $B(x, \varepsilon)$ encloses at least one point of $E \cap F$,*
- *the ball $B(x, 2\varepsilon)$ encloses at most κ points of $E \cap F$ for some fixed κ.*

In the rest of the paper, E denotes a (ε, κ)-sample of S and we provide asymptotic results when the sampling density increases, i.e. when ε tends to 0. As already mentionned, we consider κ and the surface S (and, in particular, the three quantities C, A and L) to be fixed and not to depend on ε.

Several related sampling conditions have been proposed. Amenta and Bern have introduced ε-samples [3] that fit locally the surface shape : the point density is high where the surface has high curvature or where the object or its complement is thin. However this definition is not appropriate for polyhedral surfaces since an ε-sample, as defined in [3], should have infinitely many points.

Erickson has introduced a notion of uniform sample that is related to ours but forbids two points to be too close [16]. Differently, our definition of a (ε, κ)-sample does not impose any lower bound on the minimal distance between two sample points.

In [5], Attali and Boissonnat use a slightly different definition of a (ε, κ)-sample. They assumed that for every point $x \in S$, the ball $B(x, \varepsilon)$ encloses at least one sample point and the ball $B(x, r)$ encloses $O(\frac{r^2}{\varepsilon^2})$ sample points. With this sampling condition, they proved that the complexity of the Delaunay triangulation is $O(n^{1.8})$ for general polyhedral surfaces and $O(n^{1.5})$ for convex polyhedral surfaces. Our definition of a (ε, κ)-sample is slightly more restrictive since the facets need to be sampled independently of one another, which leads to add a few more sample points near the edges. However, the two conditions are essentially the same and our linear bound holds also under the slightly more general sampling condition of [5].

Golin and Na [18] assume that the sample points are chosen uniformly at random on the surface. Such samples are (ε, κ)-samples with high probability.

3 Preliminary results

Let $n(R) = |E \cap R|$ be the number of sample points in the region $R \subset S$. Let $n = |E|$ be the total number of sample points. We first establish two propositions relating $n(R)$ and n. We start with the following lemma:

Lemma 3 *Let F be a facet of S. For any $R \subset F$, we have:*

$$\frac{a(R)}{4\pi\varepsilon^2} \leq n(R) \leq \frac{\kappa a(R \oplus \varepsilon)}{\pi\varepsilon^2}$$

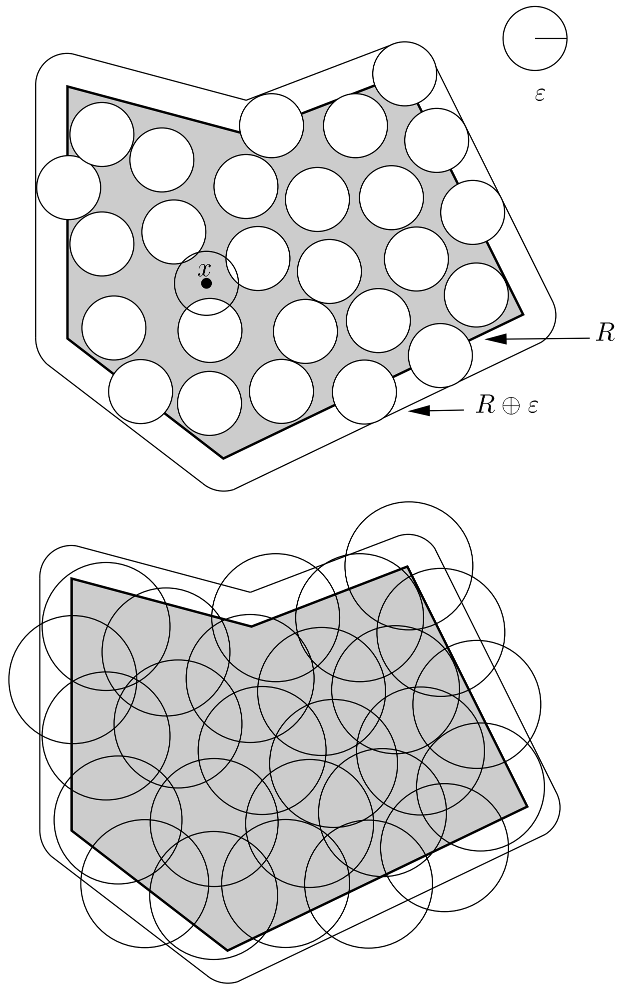

Figure 1: A maximal set of non-intersecting disks contained in $R \oplus \varepsilon$ and the corresponding covering of R obtained by doubling the radii of the disks.

Proof. Let $\cup_{i=1}^{\lambda} D(m_i, \varepsilon)$ be a maximal set of λ non-intersecting disks lying inside $R \oplus \varepsilon$. Because the set of disks is maximal, no other disk can be added without intersecting one of the λ disks $D(m_i, \varepsilon)$. This implies that no point x of R is at distance greater than 2ε from a point m_i (see Figure 1). Therefore, $\cup_{i=1}^{\lambda} D(m_i, \varepsilon)$ is a packing of $R \oplus \varepsilon$ and $\cup_{i=1}^{\lambda} D(m_i, 2\varepsilon)$ is a covering of R. Consequently:

$$\frac{a(R)}{4} \leq \lambda\pi\varepsilon^2 \leq a(R \oplus \varepsilon)$$

The disks lie in $R \oplus \varepsilon$. Therefore, the centers of the disks lie in R. By assumption, the disk $D(m_i, \varepsilon)$ contains at least one sample point and the disk $D(m_i, 2\varepsilon)$ contains at most κ sample points. Hence :

$$\lambda \leq n(R) \leq \kappa\lambda$$

and

$$\frac{a(R)}{4\pi\varepsilon^2} \leq n(R) \leq \frac{\kappa a(R \oplus \varepsilon)}{\pi\varepsilon^2}$$

□

Proposition 4 *Let F be a facet of S. For any $R \subset F$, we have:*

$$n(R) \leq 4\kappa \, \frac{a(R \oplus \varepsilon)}{A} \, n$$

Proof. We first apply Lemma 3 to bound n from below. Summing over the facets of S, we get :

$$\frac{A}{4\pi\varepsilon^2} \leq n \tag{1}$$

We apply again Lemma 3 to bound $n(R)$ from above.

$$n(R) \leq \frac{\kappa a(R \oplus \varepsilon)}{\pi\varepsilon^2}$$

Eliminating ε from the two inequalities yields the result. □

Proposition 5 *Let F be a facet of S. Let $\Gamma \subset F$ be a curve contained in F. Let $k > 0$. We have:*

$$n(\Gamma \oplus k\varepsilon) \leq \frac{(2k+1)^2}{k} \, \kappa \, \frac{l(\Gamma)}{\varepsilon} \leq \frac{2(2k+1)^2}{k} \, \sqrt{\pi} \, \kappa \, \frac{l(\Gamma)}{\sqrt{A}} \, \sqrt{n}$$

Proof. Arguing as in the proof of Lemma 3, we see that the region $\Gamma \oplus k\varepsilon$ can be covered by $\frac{l(\Gamma)}{k\varepsilon}$ disks of radius $2k\varepsilon$ centered on Γ and contained in the supporting plane of F.

Applying Lemma 3 to a disk R with radius $2k\varepsilon$, we get:

$$n(R) \leq \frac{\kappa\pi(2k\varepsilon + \varepsilon)^2}{\pi\varepsilon^2} = (2k+1)^2\kappa$$

Therefore, we have :

$$n(\Gamma \oplus k\varepsilon) \leq (2k+1)^2 \, \kappa \, \frac{l(\Gamma)}{k\varepsilon}$$

From Equation 1, we get:

$$\frac{1}{\varepsilon} \leq \frac{2\sqrt{\pi}}{\sqrt{A}} \, \sqrt{n}$$

Combining the two inequalities leads to the result. □

Lemma 6 *Let x be a sample point in the ε-regular zone of S. Let P be the supporting plane of the facet through x. Any empty sphere passing through x intersects P in a circle whose radius is less than ε.*

Proof. The proof is by contradiction. Let P be the supporting plane of F. Consider an empty sphere Σ passing through x and intersecting P along a circle of radius greater than ε (see Figure 2). Let c be the center of this circle. Let y be the point on the segment $[xc]$ at distance ε from x. Because x belongs to the ε-regular zone of F, $y \in F$. The empty sphere Σ encloses the disk $D_p(y, \varepsilon)$. Therefore, $D_p(y, \varepsilon)$ is an empty disk of P, centered on F and of radius ε, which contradicts our assumption. □

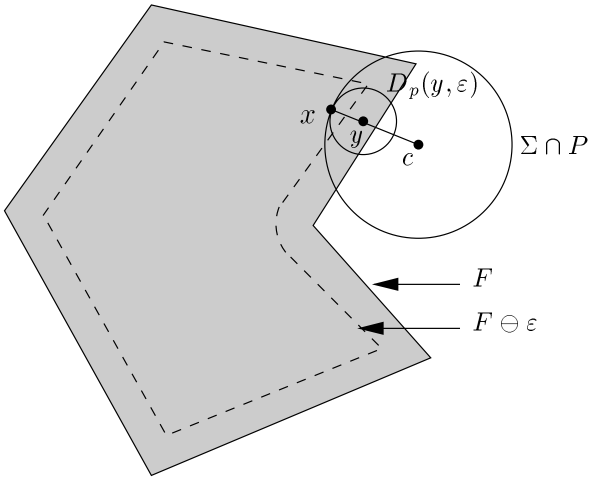

Figure 2: Assume Σ is an empty sphere passing through a point $x \in F \ominus \varepsilon$ and intersecting the supporting plane of F in a circle of radius greater than ε. Then, Σ contains an empty disk $D_p(y, \varepsilon)$ centered on F.

4 Counting Delaunay edges

Let S be a polyhedral surface and E be a (ε, κ)-sample of S. A ball or a disk is said to be empty iff its interior contains no point of E. We also say that a sphere is *empty* if the associated ball is empty. It is well-known that the Delaunay triangulation of E connects two points $p, q \in E$ iff there exists an empty sphere passing through p and q. The edge connecting p and q is called a *Delaunay edge*. We will also say that p and q are *Delaunay neighbours*.

The number of edges e_p and the number of tetrahedra t_p incident to a vertex p lying in the interior of the convex hull of E are related by Euler formula

$$t_p = 2e_p - 4$$

since the boundary of those tetrahedra is a simplicial polyhedron of genus 0. Using the same argument, if p lies on the boundary of the convex hull, we have:

$$t_p < 2e_p - 4$$

By summing over the n vertices, and observing that a tetrahedron has four vertices and an edge two, we get

$$t < e - n.$$

To bound the complexity of the Delaunay triangulation, it is therefore sufficient to count the Delaunay edges of E.

We distinguish three types of Delaunay edges : those with both endpoints in the ε-regular zone, those with both endpoints in the ε-singular zone and those with an endpoint in the ε-regular zone and the other in the ε-singular zone. They are counted separately in the following subsections,

We denote by E_s the set of sample points in the ε-singular zone of S.

4.1 Delaunay edges with both endpoints in the ε-regular zone

Lemma 7 *Let x be a sample point in the ε-regular zone and F the facet that contains x. x has at most κ Delaunay neighbours in F.*

Proof. By Lemma 6, any empty sphere passing through x intersects F in a circle whose radius is less than ε. Therefore, the

Delaunay neighbours of x on F are at distance at most 2ε from x. By assumption, the disk centered at x with radius 2ε contains at most κ points of E. □

Lemma 8 *Let x be a sample point in the ε-regular zone of a facet F. Let $F' \neq F$ be another facet of S. x has at most κ Delaunay neighbours in the ε-regular zone of facet F'.*

Proof. Refer to Figure 3. P and P' are the supporting planes of F and F', y' is a Delaunay neighbour of x in the ε-regular zone of F' and Σ is an empty sphere passing through x and y'. Σ intersects the planes P and P' along two circles whose radii are respectively r and r'. By Lemma 6, $r \leq \varepsilon$ and $r' \leq \varepsilon$.

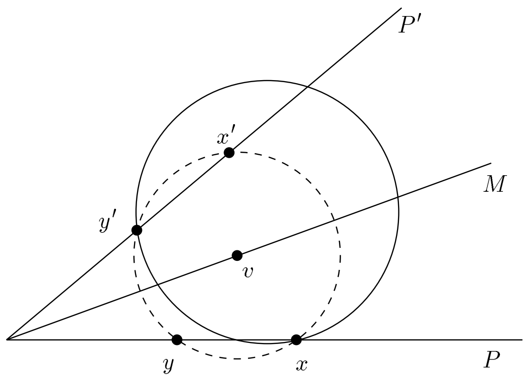

Figure 3: Any sphere passing through x and y' intersects one of the two planes P or P' in a circle whose diameter is at least $\|x' - y'\|$.

Let M be the bisector plane of P and P'. Let x' and y be the points symetric to x and y' with respect to M. Consider the smallest sphere passing through x, x', y and y'. This sphere intersects P and P' in two disks of the same radius $r_{\min}$. We claim that $r_{\min} \leq \max(r, r')$. Indeed, the Voronoi diagram of the four points x, x', y, y' consists of four unbounded regions $V(x)$, $V(x')$, $V(y')$ and $V(y)$, and four facets $V(x) \cap V(x')$, $V(x') \cap V(y')$, $V(y') \cap V(y)$ and $V(y) \cap V(x)$, all parallel to $P \cap P'$. It follows that the center c of Σ which lies in the bisector plane of x and y' must belong either to $V(x')$ or to $V(y)$. In the first case, Σ encloses x' and therefore $r_{\min} \leq r'$ while in the second it encloses y and $r_{\min} \leq r$.

We therefore have :

$$\frac{\|x' - y'\|}{2} = r_{\min} \leq \max(r, r') \leq \varepsilon$$

and consequently:

$$\|x' - y'\| \leq 2\varepsilon.$$

The Delaunay neighbours of x in the ε-regular zone of F' lie in the disk $D_{p'}(x', 2\varepsilon)$. This disk contains at most κ points of E. □

Proposition 9 *There are at most $\frac{C}{2}\kappa n$ Delaunay edges with both endpoints in the ε-regular zone of S.*

Proof. The surface has C facets. Therefore, by Lemmas 7 and 8, a point x in the ε-regular zone of S has at most $C\kappa$ Delaunay neighbours. □

4.2 Delaunay edges with both endpoints in the ε-singular zone

Proposition 10 *The number of Delaunay edges with both endpoints in the ε-singular zone is less than*

$$\frac{1}{2} 18^2 \pi \kappa^2 \frac{L^2}{A} n$$

Proof. By Proposition 5, the number $|E_s|$ of sample points in the ε-singular zone is at most

$$18 \sqrt{\pi}\, \kappa \frac{L}{\sqrt{A}} \sqrt{n}$$

Hence, the number of Delaunay edges in the ε-singular zone is at most $\frac{1}{2}|E_s| \times (|E_s| - 1) < \frac{1}{2}|E_s|^2$. □

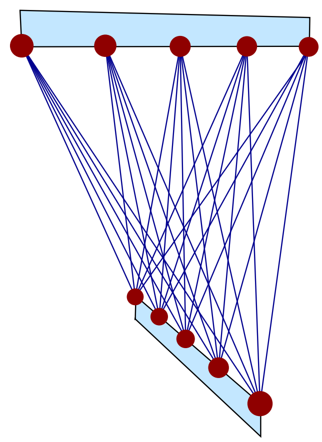

Figure 4: Example of a Delaunay triangulation of m points having a quadratic number of edges. Even if such a configuration can occur for a subset of the sample points, the number of Delaunay edges involved in this configuration is $O(n)$.

4.3 Delaunay edges joining the ε-regular and the ε-singular zones

In this section, we count the Delaunay edges with one endpoint in the ε-regular zone and the other in the ε-singular zone.

We first introduce a geometric construction of independent interest that will be useful.

Let P be a plane and E_s be a set of points. We assign to each point x of E_s the region $V(x) \subset P$ consisting of the points $p \in P$ for which the sphere tangent to P at p and passing through x encloses no point of E_s (see Figure 5). In other words, if $R(p, x)$ denotes the radius of the sphere tangent to P at p and passing through x, we have:

$$V(x) = \{p \in P : \forall y \in E_s,\ R(p, x) \leq R(p, y)\}.$$

It is easy to see that the set of all $V(x)$, $x \in E_s$, is a subdivision of P we note $\mathcal{V}$ (see Figure 8). Let $\mathcal{P}_x$ be the paraboloid of revolution with focus x and director plane P. The paraboloid $\mathcal{P}_x$ consists of the centers of the spheres passing through x and tangent to P. Assume that the points E_s are all located above plane P. If not, we replace x by the point symetric to x with respect to P, which does not change $\mathcal{V}$. Let us consider the lower envelope of the collection of paraboloids $\{\mathcal{P}_x\}_{x \in E_s}$. Cell $V(x)$ is the projection of the portion of the lower envelope contributed by $\mathcal{P}_x$ (see Figures 5 and 8).

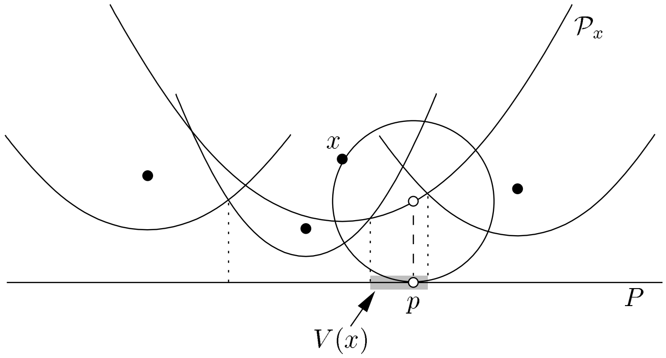

Figure 5: The cell $V(x)$ is the set of contact points between a plane P and a sphere passing through x and tangent to P. The part of the paraboloid $\mathcal{P}_x$ on the lower envelope of the paraboloids projects to the cell $V(x)$.

Consider the bisector $V(x, y)$ of $x, y \in E_s$, i.e. the points $p \in P$ such that $R(p, x) = R(p, y)$. $V(x, y)$ is the projection on P of the intersection of the paraboloids $\mathcal{P}_x$ and $\mathcal{P}_y$. Since the intersection of two paraboloids is an ellipse or a parabola, the bisector of x and y is a circle or a line (considered as a degenerated circle). Let $H(x, y)$

$$H(x, y) = \{p \in P : R(p, x) \leq R(p, y)\}.$$

Since $V(x, y)$ is a circle, $H(x, y)$ is either a disk, in which case we rename it $D(x, y)^+$, or the complementary set of a disk $D(x, y)^-$. We therefore have

$$V(x) = \bigcap_{y \in E_s, y \neq x} H(x, y) = \left(\cap D(x, y)^+\right) \setminus \left(\cup D(x, y)^-\right)$$

It follows that the edges $E(x, y)$ of $V(x)$ are circle arcs that we call *convex* or *concave* wrt x depending whether the disk $D(x, y)$ (whose boundary contains $E(x, y)$) is labelled $+$ or $-$ (see Figure 6). Observe that the convex edges of $V(x)$ are included in the boundary of the convex hull of $V(x)$.

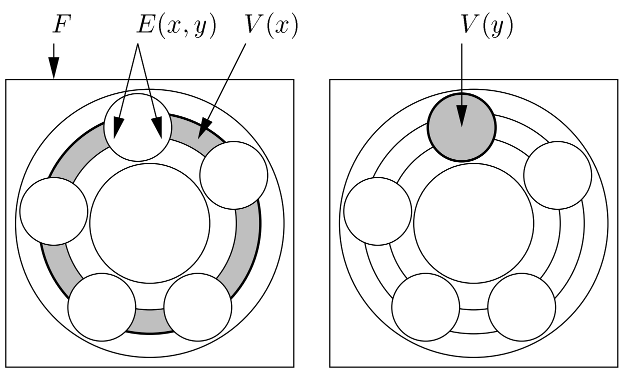

Figure 6: The bold edges are the convex edges of the shaded cells. The edge $E(x, y)$ which is concave wrt x is convex wrt y. The convex edges of a cell lie on the boundary of its convex hull.

Proposition 11 *The number of Delaunay edges with one endpoint in the ε-regular zone and the other in the ε-singular zone is at most :*

$$\left(1 + 450\,\pi\,\kappa^2\,\frac{L^2}{A}\right) n$$

Proof. Let F be a facet of S and P the supporting plane of F. We bound the number of Delaunay edges with one endpoint in E_s and the other in $E \cap (F \ominus \varepsilon)$.

We denote by $\mathcal{V}_F$ the restriction of the subdivision $\mathcal{V}$ introduced above to F, and, for $x \in E_s$, we denote by $V(x)$ the cell of $\mathcal{V}_F$ associated to x.

We first show that the Delaunay neighbours of x that belong to the ε-regular zone of F belong to $V(x) \oplus 2\varepsilon$. Consider a Delaunay edge (xf) with $x \in E_s$, $x \notin F \ominus \varepsilon$ and $f \in E_s \cap (F \ominus \varepsilon)$. Let Σ be an empty sphere passing through x and f, v its center (see Figure 7). By Lemma 6, Σ intersects P in a circle whose radius r is less than ε. For a point c on the segment $[vx]$, we note Σ_c the sphere centered at c and passing through x. Because Σ encloses Σ_c, Σ_c is an empty sphere. For $c = v$, Σ_c intersects P. For $c = x$, Σ_c does not intersect P. Consequently, there exists a position of c on $[vx]$ for which Σ_c is tangent to P. Let $p = \Sigma_c \cap P$ for such a point c. We have $p \in V(x)$ and $\|p - f\| \leq 2r \leq 2\varepsilon$. Hence, $f \in V(x) \oplus 2\varepsilon$. Let (xf) be a Delaunay edge with $x, f \in E_s \cap (F \ominus \varepsilon)$. Applying Lemma 6 leads to $f \in V(x) \oplus 2\varepsilon$.

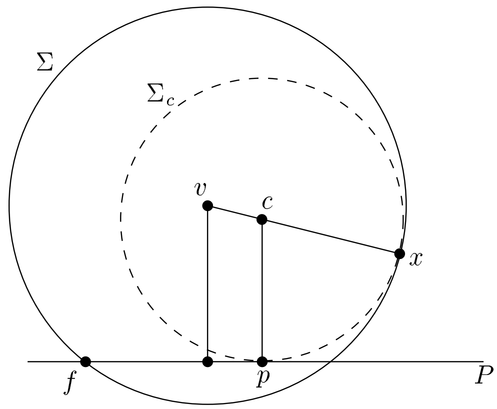

Figure 7: Every sphere Σ passing through x and $f \in P$ contains a sphere Σ_c passing through x and tangent to P.

Let N_F be the number of Delaunay edges between E_s and $F \ominus \varepsilon$. We have, using the fact that $\mathcal{V}_F$ is a subdivision of F and Proposition 5 :

$$\begin{aligned} N_F &\leq \sum_{x \in E_s} n(V(x) \oplus 2\varepsilon) \\ &\leq n(F) + \sum_{x \in E_s} n(\partial V(x) \oplus 2\varepsilon) \\ &\leq n(F) + 25\sqrt{\pi}\,\kappa\,\frac{1}{\sqrt{A}}\,\sqrt{n}\,\sum_{x \in E_s} l(\partial V(x)) \end{aligned}$$

Let us bound $\sum_{x \in E_s} l(\partial V(x))$. Given a cell $V(x)$, we bound the length of its convex edges. By summing over all $x \in E_s$, all edges in $\mathcal{V}_F$ will be taken into account.

The convex edges of x are contained in the boundary of the convex hull of $V(x)$. Since $V(x) \subset F$, the length of the boundary of the convex hull of $V(x)$ is at most the length of ∂F. Consequently:

$$\sum_{x \in E_s} l(\partial V(x)) \leq l(\partial F) \times |E_s|$$

Since, by Proposition 5, $|E_s| \leq 18\sqrt{\pi}\,\kappa\,\frac{L}{\sqrt{A}}\,\sqrt{n}$, we have:

$$N_F \leq n(F) + 450\,\pi\,\kappa^2\,\frac{l(\partial F) \times L}{A}\,n$$

By summing over all the facets, we conclude that the total number of Delaunay edges with one endpoint in the ε-regular zone and the other in ε-singular zone is at most :

$$\left(1 + 450\,\pi\,\kappa^2\,\frac{L^2}{A}\right) n$$

□

4.4 Main result

We sum up our results in the following theorem :

Theorem 12 *Let S be a polyhedral surface and E a (ε, κ)-sample of S of size $|E| = n$. The number of edges in the Delaunay triangulation of E is at most :*

$$\left(1 + \frac{C\,\kappa}{2} + 612\,\pi\,\kappa^2\,\frac{L^2}{A}\right) n$$

5 Conclusion

We have shown that, under a mild sampling condition, the Delaunay triangulation of points scattered over a fixed polyhedral surface or any fixed pure piece-wise linear complex has linear complexity. This result corroborates the experimental observation that, in most practical cases, the triangulation is linear (see, e.g., [8]). Our sampling condition is less restrictive than Erickson's one [16] and is fulfilled with high probability if the sample points are chosen uniformly at random as suggested by Golin and Na [18].

In this paper, we have expressed our sampling condition in a simple and intuitive way. However, our linear bound holds under a more general setting. Indeed, all we need for the proof is a partition of the surface in two zones, an ε-regular zone where one can apply Lemma 6 and an ε-singular zone containing $O(\sqrt{n})$ points.

Observe that our linear bound can easily be extended to $(d-1)$-dimensional pure piece-wise linear complexes of $\mathbb{R}^d$ for $d > 3$.

The main open question is of course to consider the case of smooth surfaces. The $O(n\sqrt{n})$ lower bound obtained by Erickson for cylinders show that a linear bound does not hold for arbitrary surfaces. We conjecture that, for generic surfaces, the complexity of the Delaunay triangulation is still linear. We say that a surface S is generic if 1. its maximal balls intersect S at a finite number of so-called contact points, and 2. the intersection of S with the union of the maximal balls with only one contact point form a set of curves of finite length on S. In particular, generic surfaces cannot contain spherical nor cylindrical pieces.

References

[1] N. Amenta, M. Bern, and M. Kamvysselis. A new Voronoi-based surface reconstruction algorithm. In *Proc. SIGGRAPH '98*, pages 415–412, July 1998.

[2] N. Amenta, S. Choi, T.K. Dey, and N. Leekha. A simple algorithm for homeomorphic surface reconstruction. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 213–222, 2000.

[3] Nina Amenta and Marshall Bern. Surface reconstruction by voronoi filtering. *Discrete Comput. Geom.*, 22:481–504, 1999.

[4] Nina Amenta and Ravi Krishna Kolluri. Accurate and efficient unions of balls. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 119–128, 2000.

[5] Dominique Attali and Jean-Daniel Boissonnat. Complexity of the delaunay triangulation of points on polyhedral surfaces. Technical Report RR-4232, INRIA, July 2001. `http://www-sop.inria.fr/rapports/sophia/RR-4232.html`.

[6] F. Bernardini, C. L. Bajaj, J. Chen, and D. R. Schikore. Automatic reconstruction of 3D CAD models from digital scans. *Internat. J. Comput. Geom. Appl.*, 9:327–369, 1999.

[7] F. Bernardini, J. Mittleman, H. Rushmeier, C. Silva, and G. Taubin. The ball-pivoting algorithm for surface reconstruction. *IEEE Trans. on Visualization and Computer Graphics*, 5(4):349–359, 1999.

[8] J.-D. Boissonnat and F. Cazals. Smooth surface reconstruction via natural neighbour interpolation of distance functions. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 223–232, 2000.

[9] Jean-Daniel Boissonnat and Frédéric Cazals. Natural neighbour coordinates of points on a surface. *Computational Geometry - Theory and Application*, 19(2-3):87–220, 2001.

[10] Jean-Daniel Boissonnat and Mariette Yvinec. *Algorithmic Geometry*. Cambridge University Press, UK, 1998. Translated by Hervé Brönnimann.

[11] T. M. Chan, J. Snoeyink, and C. K. Yap. Primal dividing and dual pruning: Output-sensitive construction of 4-d polytopes and 3-d Voronoi diagrams. *Discrete Comput. Geom.*, 18:433–454, 1997.

[12] L. P. Chew. Guaranteed-quality mesh generation for curved surfaces. In *Proc. 9th Annu. ACM Sympos. Comput. Geom.*, pages 274–280, 1993.

[13] R. A. Dwyer. Higher-dimensional voronoi diagram in linear expected time. *Discrete Comput. Geom.*, 6:343–367, 1991.

[14] R. A. Dwyer. The expected number of k-faces of a voronoi diagram. *Internat. J. Comput. Math.*, 26(5):13–21, 1993.

[15] H. Edelsbrunner. Deformable smooth surface design. *Discrete Comput. Geom.*, 21:87–115, 1999.

[16] Jeff Erickson. Nice point sets can have nasty Delaunay triangulations. In *Proc. 17th Annu. ACM Sympos. Comput. Geom.*, pages 96–105, 2001.

[17] Jeff Erickson. Dense point sets have sparse delaunay triangulations. In *Proc. of the 13th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2002. To appear. `http://www.cs.ust.hk/tcsc/RR/`.

[18] Mordecai Golin and HyeonSuk Na. On the average complexity of 3d-voronoi diagrams of random points on convex polytopes. In *Proc. 12th Canad. Conf. Comput. Geom.*, 2000. `http://www.cs.ust.hk/tcsc/RR/`.

[19] D.J. Sheehy, C.G. Armstrond, and D.J. Robinson. Computing the medial surface of a solid from a domain delaunay triangulation. In *Proc. 3rd Symposium on Solid Modeling and Applications*, pages 201–212, May 1995.

[20] Andrew Thall, Stephen Pizer, and Tom Fletcher. Deformable solid modeling using sampled medial surfaces: A multiscale approach. Technical Report TR00-005, January 2000. `http://citeseer.nj.nec.com/thall00deformable.html`.

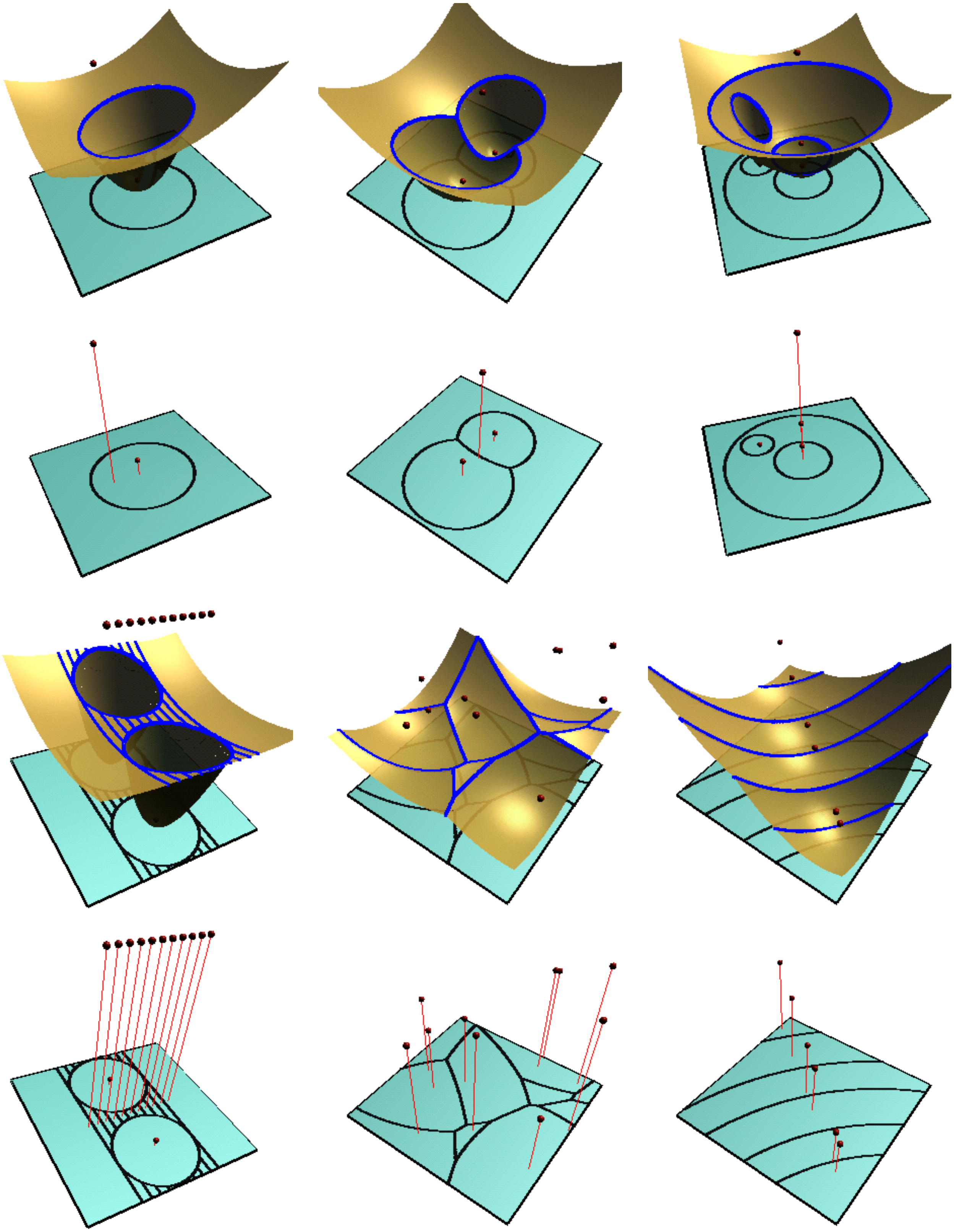

Figure 8: Decomposition of a facet F into cells for different set of points E_s. The lower envelope of the paraboloid $\{\mathcal{P}_x\}_{x \in E_s}$ has been represented. The red spheres represent the points of E_s and the red lines materialize the projection of the points of E_s on the plane P. The bisector of two points is a circle. The projection of x on P do not belong necessary to its cell. The decomposition of F can have a quadratic number of edges.