Overview

Earlier lecture
- Statistical sampling and Monte Carlo integration

Last lecture
- Signal processing view of sampling

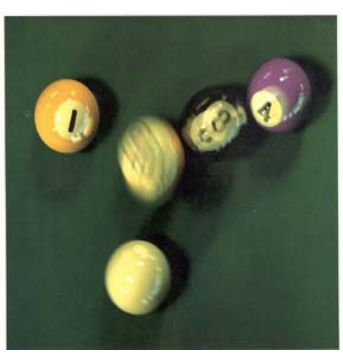
Today
- Variance reduction
- Importance sampling
- Stratified sampling
- Multidimensional sampling patterns
- Discrepancy and Quasi-Monte Carlo

Latter
- Path tracing for interreflection
- Density estimation

Cameras

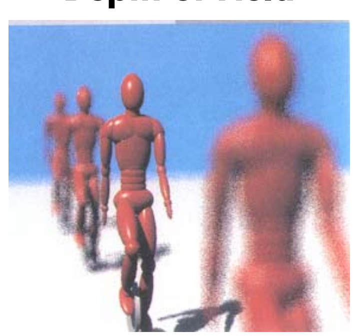
$$R = \int_T \iint_\Omega \iint_A L(x,\omega,t)\ P(x)\ S(t) \cos\theta\ dA\ d\omega\ dt$$

Motion Blur



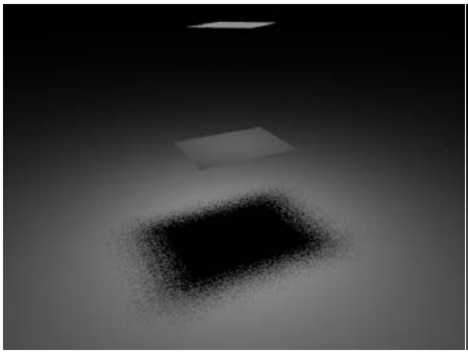
Source: Cook, Porter, Carpenter, 1984

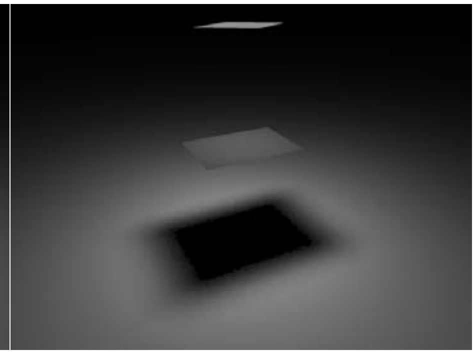
Depth of Field



Source: Mitchell, 1991

Variance





1 shadow ray per eye ray

16 shadow rays per eye ray

Variance

Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2] - E[Y]^2$$

Variance decreases with sample size

$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N^2}\sum_{i=1}^{N} V[Y_i] = \frac{1}{N^2} NV[Y] = \frac{1}{N} V[Y]$$

Variance Reduction

Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

If one technique has twice the variance as another technique, then it takes twice as many samples to achieve the same variance

If one technique has twice the cost of another technique with the same variance, then it takes twice as much time to achieve the same variance

Techniques to increase efficiency

- **Importance sampling**
- **Stratified sampling**

Biasing

Previously used a uniform probability distribution

Can use another probability distribution

$$X_i \sim p(x)$$

But must change the estimator

$$Y_i = \frac{f(X_i)}{p(X_i)}$$

Unbiased Estimate

Probability $X_i \sim p(x)$

Estimator $Y_i = \frac{f(X_i)}{p(X_i)}$

$$E[Y_i] = E\left[\frac{f(X_i)}{p(X_i)}\right]$$
$$= \int \left[\frac{f(X_i)}{p(X_i)}\right] p(x)\,dx$$
$$= \int f(x)\,dx$$
$$= I$$

Importance Sampling

Sample according to *f*

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\int \tilde{p}(x)\,dx = \int \frac{f(x)}{E[f]}\,dx$$
$$= \frac{1}{E[f]} \int f(x)\,dx$$
$$= 1$$

Importance Sampling

Variance

$$V[f] = E[f^2] - E^2[f]$$

Sample according to *f*

$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$\tilde{f}(x) = \frac{f(x)}{\tilde{p}(x)}$$

$$E[\tilde{f}^2] = \int \left[\frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x)\,dx$$

$$= \int \left[\frac{f(x)}{f(x)/E[f]} \right]^2 \frac{f(x)}{E[f]}\,dx$$

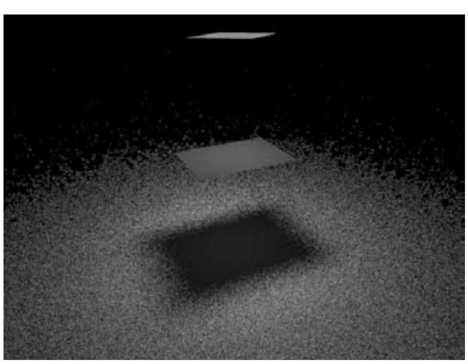
$$= E[f] \int f(x)\,dx$$

$$= E^2[f]$$

Zero variance!

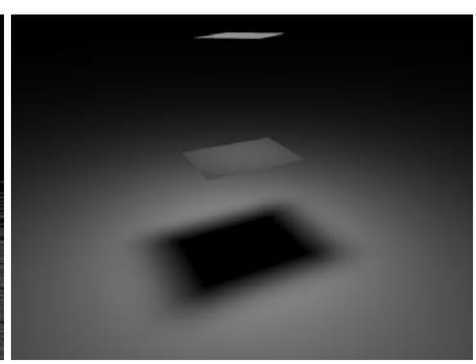
$$V[\tilde{f}^2] = 0$$

Examples



Projected solid angle

4 eye rays per pixel
100 shadow rays

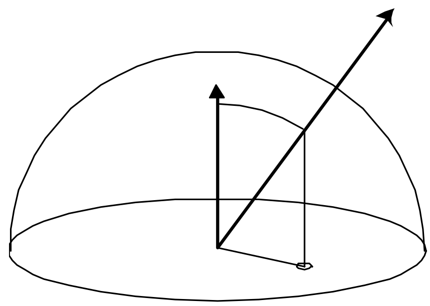


Area

4 eye rays per pixel
100 shadow rays

Irradiance

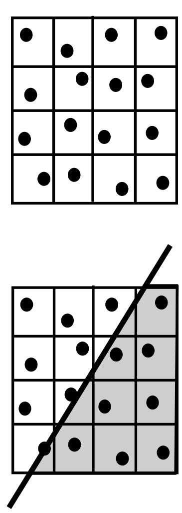
Generate cosine weighted distribution



$$p(\omega)\, d\omega = \cos\theta\, d\omega$$

$$E = \int_{H^2} L_i(\omega_i) \cos\theta_i \, d\omega_i$$

Stratified Sampling



Stratified sampling is like jittered sampling

Allocate samples per region

$$F_N = \frac{1}{N} \sum_{i=1}^{N} F_i$$

New variance

$$V[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} V[F_i]$$

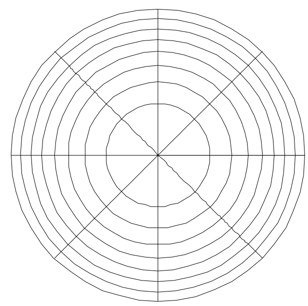
Thus, if the variance in regions is less than the overall variance, there will be a reduction in resulting variance

For example: An edge through a pixel

$$V[F_N] = \frac{1}{N^2} \sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_E]}{N^{1.5}}$$

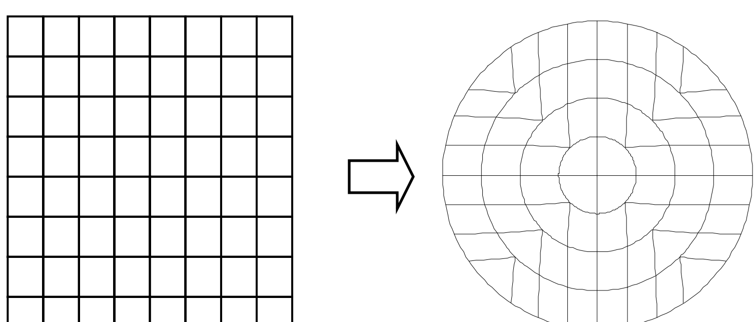
Sampling a Circle

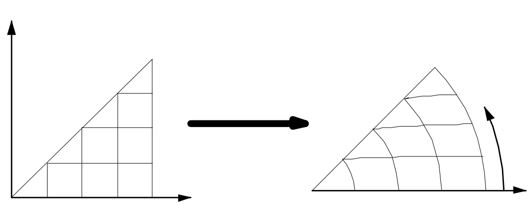
Equi-Areal



$$\theta = 2\pi U_1$$
$$r = \sqrt{U_2}$$

Shirley's Mapping





$$r = U_1$$
$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

High-dimensional Sampling

Numerical quarature

For a given error …

$$E \sim \frac{1}{n} = \frac{1}{N^{1/d}}$$

Random sampling

For a given variance …

$$E \sim V^{1/2} \sim \frac{1}{N^{1/2}}$$

Monte Carlo requires fewer samples for the same error in high dimensional spaces

Block Design

Latin Square

a	*d*	*c*	*b*
b	*a*	*d*	*c*
c	*b*	*a*	*d*
d	*c*	*b*	*a*

Alphabet of size *n*

Each symbol appears exactly once in each row and column

Rows and columns are stratified

Block Design

N-Rook Pattern

a			
		a	
	a		
			a

Incomplete block design

Replaced n^2 samples with n samples

Permutations: $(\pi_1(i), \pi_2(i), \cdots \pi_d(i))$

Generalizations: N-queens, 2D projection

$$(\pi_x = \{1,2,3,4\}, \pi_y = \{4,2,3,1\})$$

Space-time Patterns

6	10	2	13
3	14	12	8
15	0	7	11
5	9	4	1

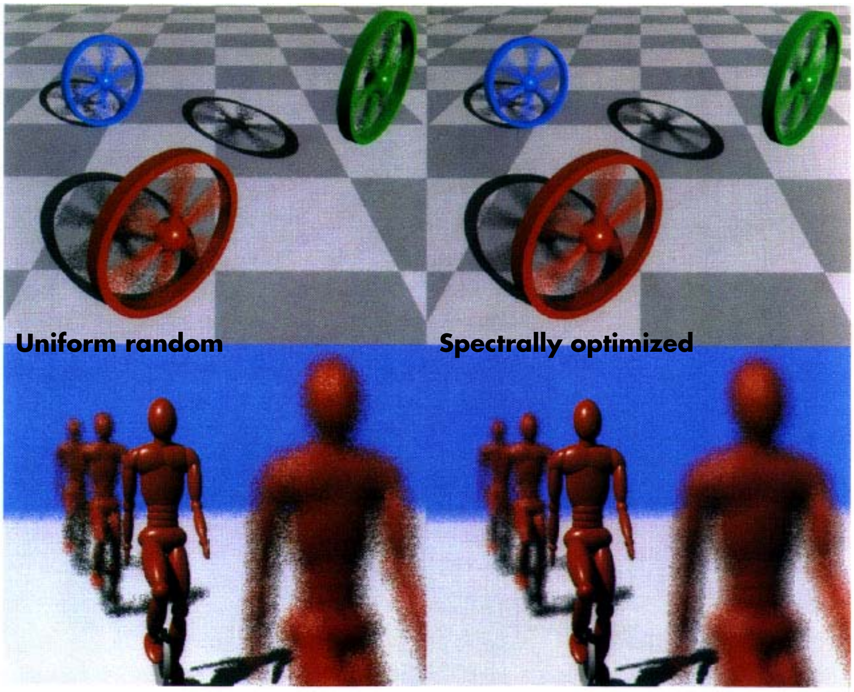
Cook Pattern

15	8	5	2
4	3	14	9
10	13	0	7
1	6	11	12

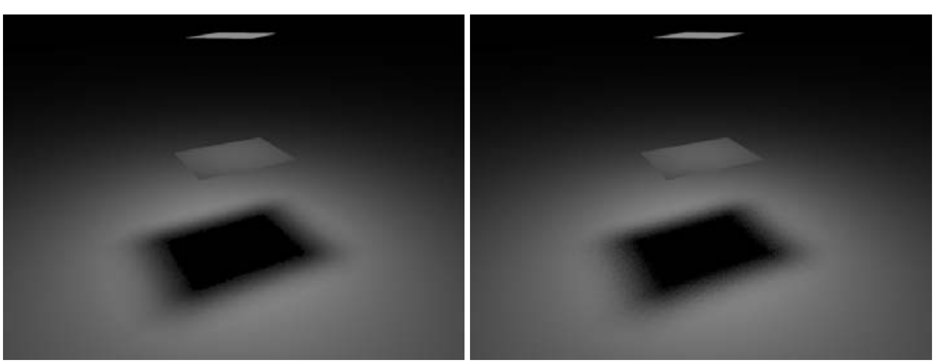
Pan-diagonal Magic Square

Distribute samples in time

- **Complete in space**
- **Samples in space should have blue-noise spectrum**
- **Incomplete in time**
- **Decorrelate space and time**
- **Nearby samples in space should differ greatly in time**



Path Tracing



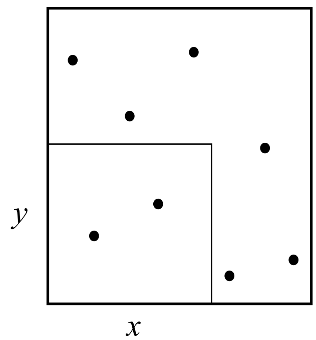
4 eye rays per pixel
16 shadow rays per eye ray

Complete

64 eye rays per pixel
1 shadow ray per eye ray

Incomplete

Discrepancy



$$\Delta(x,y) = \frac{n(x,y)}{N} - xy$$

$A = xy$

$n(x,y)$ number of samples in A

$$D_N = \max_{x,y} \left|\Delta(x,y)\right|$$

Theorem on Total Variation

Theorem:

$$\left|\frac{1}{N}\sum_{i=1}^{N} f(X_i) - \int f(x)\,dx\right| \le V(f) D_N$$

Proof: Integrate by parts

$$\int f(x) \; [\frac{\delta(x-x_i)}{N} - 1]\,dx \qquad \frac{\partial \Delta(x)}{\partial x} = \frac{\delta(x-x_i)}{N} - 1$$

$$= \int f(x) \frac{\partial \Delta(x)}{\partial x}\,dx$$

$$= f\Delta\Big|_0^1 - \int \frac{\partial f(x)}{\partial x}\Delta(x)\,dx = -\int \frac{\partial f(x)}{\partial x}\Delta(x)\,dx$$

$$\le D_N \int \left|\frac{\partial f(x)}{\partial x}\right| dx = V(f) D_N$$

CS348B Lecture 9 Pat Hanrahan, Spring 2005

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer *i* in integer base *b*

$$i = d_i \cdots d_2 d_1 d_0$$
$$\phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

$\phi_2(i)$

1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

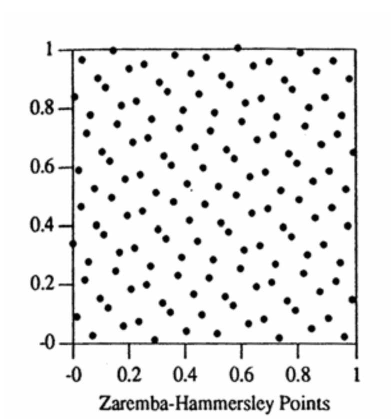
Hammersley points

$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots) \qquad D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

Halton points (sequential)

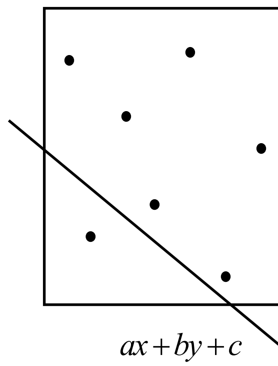
$$(\phi_2(i), \phi_3(i), \phi_5(i), \cdots) \qquad D_N = O\left(\frac{\log^{d} N}{N}\right)$$

Hammersly Points



$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

Edge Discrepancy



Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended

Views of Integration

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good
2. Statistical sampling (Monte Carlo)
   - Sampling like polling
   - Variance
   - High dimensional sampling: $1/N^{1/2}$
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Numerical
   - Quadrature/Integration rules
   - Smooth functions