Overview

Earlier lecture
- Statistical sampling and Monte Carlo integration

Last lecture
- Signal processing view of sampling

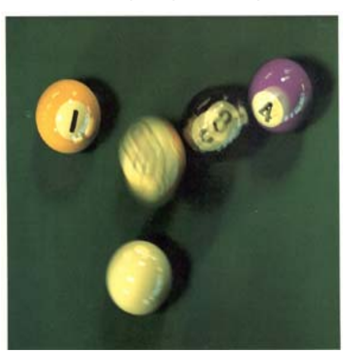
Today
- Variance reduction
- Importance sampling
- Stratified sampling
- Multidimensional sampling patterns
- Discrepancy and Quasi-Monte Carlo

Latter
- Path tracing for interreflection
- Density estimation

Cameras

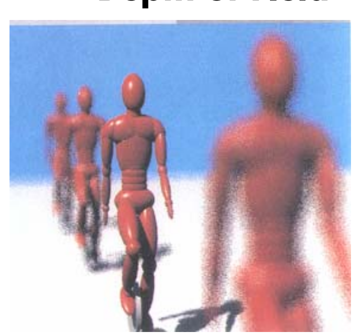
$$R = \int_T \iint_\Omega \iint_A L(x,\omega,t)\, P(x) S(t) \cos\theta \, dA \, d\omega \, dt$$

Motion Blur



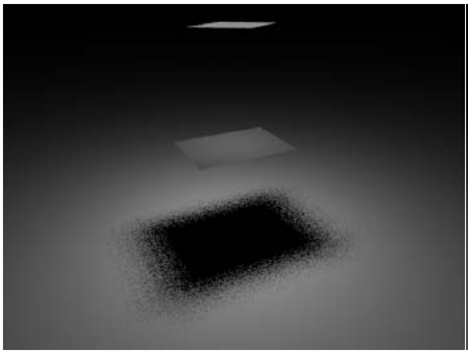
Source: Cook, Porter, Carpenter, 1984

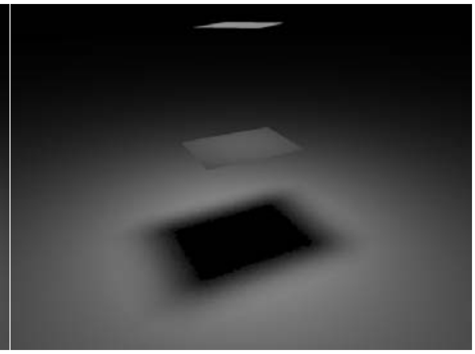
Depth of FIeld



Source: Mitchell, 1991

Variance





4 eye rays per pixel
1 shadow ray per eye ray

4 eye rays per pixel
16 shadow rays per eye ray

Variance

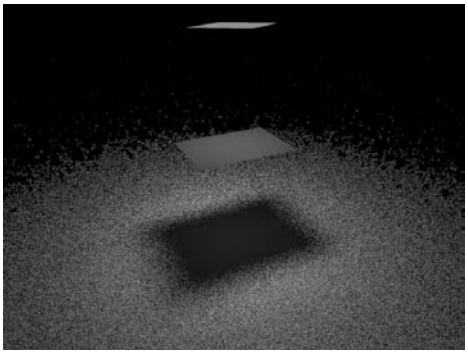
Definition

$$V[Y] \equiv E[(Y - E[Y])^2]$$
$$= E[Y^2] - E[Y]^2$$

Variance decreases with sample size

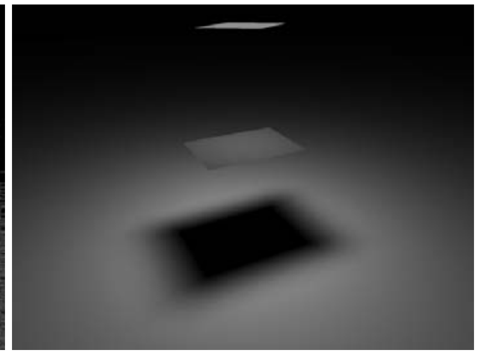
$$V[\frac{1}{N}\sum_{i=1}^{N} Y_i] = \frac{1}{N} V[Y]$$

Examples



Projected solid angle

4 eye rays per pixel
100 shadow rays



Area

4 eye rays per pixel
100 shadow rays

Variance Reduction

Efficiency measure

$$Efficiency \propto \frac{1}{Variance \bullet Cost}$$

Some techniques

- Estimators
- Expected values vs. rejection sampling
- Importance sampling
- Sampling patterns: stratified, correlated, antithetic

CS348B Lecture 9 Pat Hanrahan, Spring 2004

Biasing

Biasing the sampling process

$$X_i \sim p(x) \quad Y_i = \frac{f(X_i)}{p(X_i)}$$

$$\begin{aligned} E[Y_i] &= E\left[\frac{f(X_i)}{p(X_i)}\right] \\ &= \int \left[\frac{f(X_i)}{p(X_i)}\right] p(x)\,dx \\ &= \int f(x)\,dx \\ &= I \end{aligned}$$

Importance Sampling

Variance

$$V[f] = E[f^2] - E^2[f] \qquad E[Y_i^2] = \int \left[\frac{f(X_i)}{p(X_i)}\right]^2 p(x)\,dx$$

Zero variance biasing

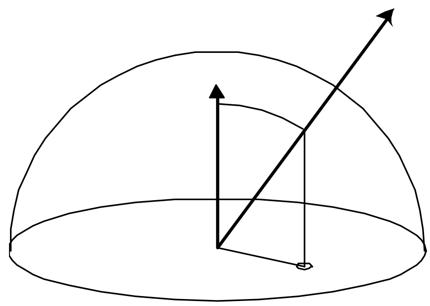
$$\tilde{p}(x) = \frac{f(x)}{E[f]}$$

$$V[\tilde{f}^2] = 0$$

$$\begin{aligned} E[\tilde{f}^2] &= \int \left[\frac{f(x)}{\tilde{p}(x)}\right]^2 \tilde{p}(x)\,dx \\ &= \int \left[\frac{f(x)}{f(x)/E[f]}\right]^2 \frac{f(x)}{E[f]}\,dx \\ &= E^2[f] \end{aligned}$$

Irradiance

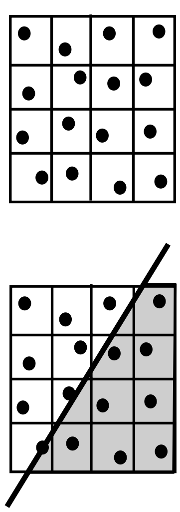
Generate cosine weighted distribution



$$p(\omega)\,d\omega = \cos\theta\,d\omega$$

$$E = \int_{H^2} L_i(\omega_i)\cos\theta_i\,d\omega_i$$

Stratified Sampling



Stratified sampling like jittered sampling

Allocate samples per region

$$N = \sum_{i=1}^{m} N_i \quad F_N = \frac{1}{N}\sum_{i=1}^{m} N_i F_i$$

New variance

$$V[F_N] = \frac{1}{N^2}\sum_{i=1}^{N} N_i V[F_i]$$

Thus, if the variance in regions is less than the overall variance, there will be a reduction in resulting variance

For example: An edge through a pixel

$$V[F_N] = \frac{1}{N^2}\sum_{i=1}^{\sqrt{N}} V[F_j] = \frac{V[F_E]}{N^{1.5}}$$

High-dimensional Sampling

Stratified sampling (also numerical quadrature)

For a given error …

$$E \sim \frac{1}{N^d}$$

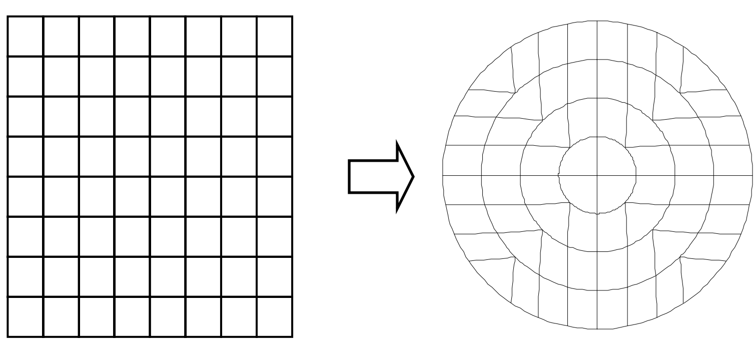
Random sampling

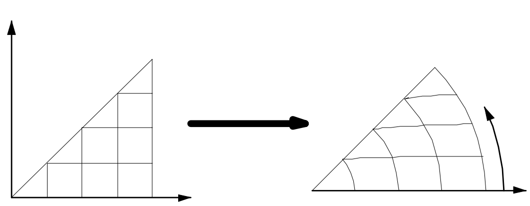
For a given variance …

$$E \sim V^{1/2} \sim \frac{1}{N^{1/2}}$$

Monte Carlo much better for Integration in high dimensional spaces

Shirley's Mapping





$$r = U_1$$

$$\theta = \frac{\pi}{4}\frac{U_2}{U_1}$$

Block Design

a	d	c	b
b	a	d	c
c	b	a	d
d	c	b	a

Latin Square

Alphabet of size *n*

Each symbol appears exactly once in each row and column

Improves discrepancy

a			
		a	
	a		
			a

N-Rook Pattern

$(\pi_x = \{1,2,3,4\}, \pi_y = \{4,2,3,1\})$

Incomplete block design

Replaced N^d samples with N samples

Permutations: $(\pi_1(i), \pi_2(i), \cdots \pi_d(i))$

Generalizations: N-queens, 2D projection

Space-time Patterns

6	10	2	13
3	14	12	8
15	0	7	11
5	9	4	1

Cook Pattern

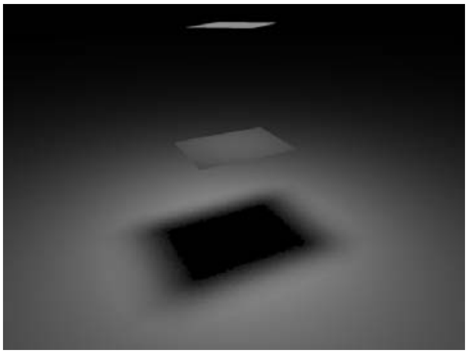
15	8	5	2
4	3	14	9
10	13	0	7
1	6	11	12

Pan-diagonal Magic Square

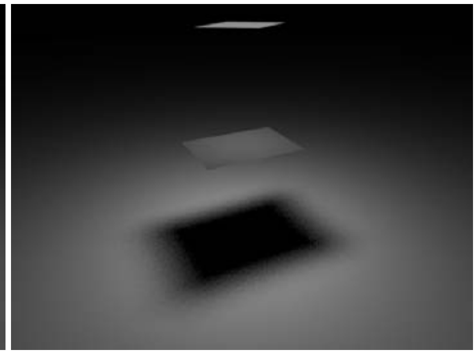
Distribute *t* samples

- **Decorrelate space and time**
- **Nearby samples in space should differ greatly in time**

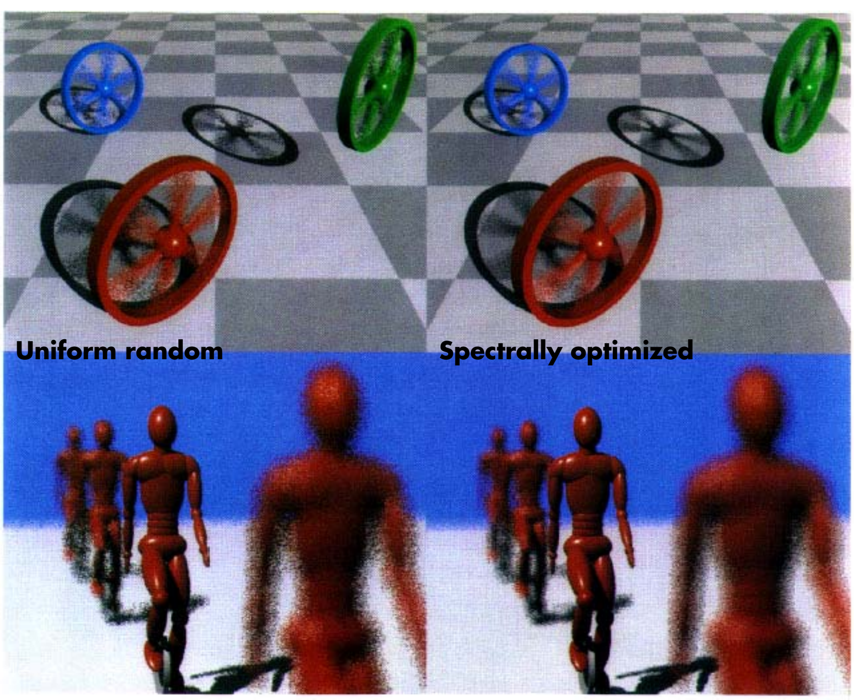
Examples



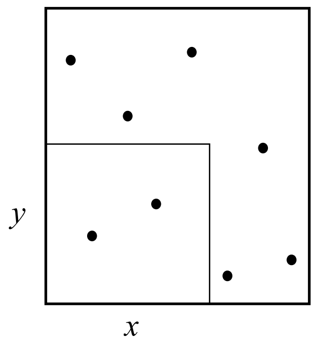
4 eye rays per pixel
16 shadow rays per eye ray



64 eye rays per pixel
1 shadow ray per eye ray



Discrepancy



$$\Delta(x,y) = \frac{n(x,y)}{N} - xy$$

$A = xy$

$n(x,y)$ number of samples in A

$$D_N = \max_{x,y} \left|\Delta(x,y)\right|$$

Theorem on Total Variation

Theorem:

$$\left|\frac{1}{N}\sum_{i=1}^{N} f(X_i) - \int f(x)\,dx\right| \le V(f) D_N$$

Proof: Integrate by parts

$$\int f(x) \; [\frac{\delta(x-x_i)}{N} - 1]\,dx \qquad \frac{\partial\Delta(x)}{\partial x} = \frac{\delta(x-x_i)}{N} - 1$$

$$= \int f(x)\frac{\partial\Delta(x)}{\partial x}\,dx$$

$$= f\Delta\Big|_0^1 - \int \frac{\partial f(x)}{\partial x}\Delta(x)\cdots dx = -\int \frac{\partial f(x)}{\partial x}\Delta(x)\,dx$$

$$\le D_N \int \left|\frac{\partial f(x)}{\partial x}\right| dx = V(f) D_N$$

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer *i* in integer base *b*

$$i = d_i \cdots d_2 d_1 d_0$$

$$\phi_b(i) \equiv 0.d_0 d_1 d_2 \cdots d_i$$

			$\phi_2(i)$
1	1	.1	1/2
2	10	.01	1/4
3	11	.11	3/4
4	100	.001	3/8
5	101	.101	5/8

Hammersley points

$$(i/N, \phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

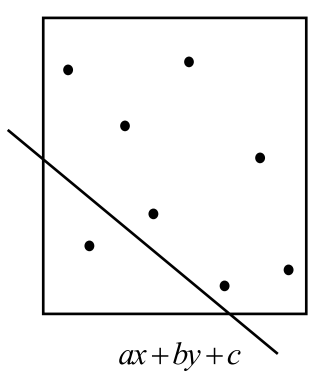
$$D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

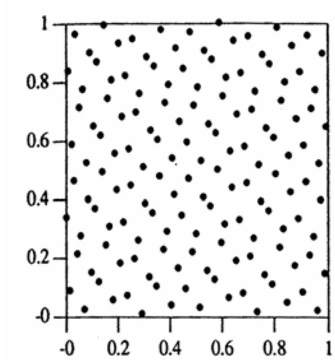
Halton points (sequential)

$$(\phi_2(i), \phi_3(i), \phi_5(i), \cdots)$$

$$D_N = O\left(\frac{\log^{d} N}{N}\right)$$

Edge Discrepancy





Zaremba-Hammersley Points

Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples

Low-Discrepancy Patterns

Process	16 points	256 points	1600 points
Zaremba	0.0504	0.00478	0.00111
Jittered	0.0538	0.00595	0.00146
Poisson-Disk	0.0613	0.00767	0.00241
N-Rooks	0.0637	0.0123	0.00488
Random	0.0924	0.0224	0.00866

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended

Views of Integration

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good
2. Statistical sampling
   - Monte Carlo: variance, central limit theorem
   - Adaptive sampling criteria
   - $N^{-1/2}$ - high dimensional sampling
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Numerical
   - Quadrature/Integration rules
   - Smooth functions