Homework #2: Curves and their polar forms [65 points]
Due Date: Monday, 11 February 2008

**Problem 1. [15 points]**

Consider the curves $Y = X^2$, $Y = X^3$, and $Y^2 = X^3$ in the neighborhood of the origin. In all three cases, the line $Y = 0$ is the tangent line to the curve at the origin, but the flavors of tangency are different. The line $Y = 0$ is a *simple tangent* to the curve $Y = X^2$ at the origin, an *inflectional tangent* to the curve $Y = X^3$ at the origin, and a *cuspidal tangent* to the curve $Y^2 = X^3$ at the origin. Sketch the three situations.

Let $A := [0;1,0]$ denote the point at infinity in the horizontal direction. Find a curve for which the line $Y = 0$ is a simple tangent at the point $A$. Similarly, find a curve for which the line $Y = 0$ is an inflectional tangent at $A$ and one for which the line $Y = 0$ is a cuspidal tangent at $A$. Give the equations of your three curves and sketch them.

Let $B := [0;0,1]$ denote the point at infinity in the vertical direction. Find a curve for which the line at infinity is a simple tangent at the point $B$, a curve for which the line at infinity is an inflectional tangent at $B$, and a curve for which the line at infinity is a cuspidal tangent at $B$. Once again, give the equations of your three curves and sketch them.

Hint: Simple tangents are preserved by projective maps; that is, if a line $\ell$ is a simple tangent to a curve $C$ at a point $P$, and if $F$ is any invertible, projective map, then the line $F(\ell)$ is a simple tangent to the curve $F(C)$ at the point $F(P)$. Inflectional tangents and cuspidal tangents are also preserved by projective maps. One simple family of projective maps are those that simply permute the three homogeneous coordinates $w$, $x$, and $y$.

**Problem 2. [15 points]**

Choose three points in the plane, label them $f(0,0)$, $f(0,3)$, and $f(3,3)$, and sketch the parabolic segment $F([0\,..\,3])$ that has those three points as its Bézier control points and $f$ as its polar form. Add points and lines to your sketch so as to construct $F(1)$ and $F(2)$, and label whatever new points you have added as polar values of $F$. Redraw your sketch, if necessary, to make it neat and clear. Considering the resulting figure, give two different recipes for constructing the point $f(1,2)$ from other polar values. Why do these two recipes give the same answer? What does this have to do with the medians of the triangle whose vertices are $f(0,1)$, $f(0,3)$, and $f(2,3)$?

**Problem 3. [20 points]**

Let $F([0\,..\,1])$ be the cubic segment in the plane whose four Bézier control points are $f(0,0,0) := (0,0)$, $f(0,0,1) := (\lambda,\lambda)$, $f(0,1,1) := (1-\lambda,\lambda)$, and $f(1,1,1) := (1,0)$, where $\lambda$ is some positive number. For each positive $\lambda$, classify the cubic $F$ as either humpy, pointy, loopy, or

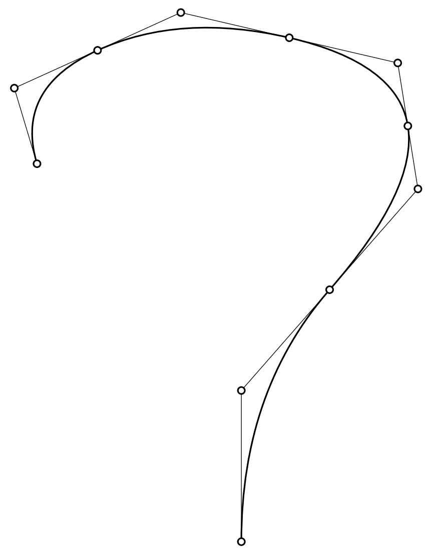

Figure 1: A quadratic spline with one non-joint

parabolic. For which of the $\lambda$ that make $F$ humpy do the two points of inflection on $F$ correspond to times $t$ in the interval $[0\,..\,1]$? For which of the $\lambda$ that make $F$ loopy do the two times $t$ that correspond to $F$'s self-intersection lie in the interval $[0\,..\,1]$?

Hints: Compute the $X$ and $Y$ coordinates of $F(t)$, say $X(t,\lambda)$ and $Y(t,\lambda)$. An inflection point, called a *flex* for short, is a point where the velocity and acceleration vectors are parallel. A self-intersection, called a *crunode*, is a single point that corresponds to two different times: $F(t_1) = F(t_2)$. A humpy cubic has two real, finite flexes, but no crunode. A loopy cubic has a crunode, but no real, finite flexes.

**Problem 4. [15 points]**

Figure 1 shows a quadratic, polynomial spline curve $F$ on the knot sequence

$$(0,0,1,2,3,4,5,5),$$

of length 8. The spline $F$ looks something like a question mark. The upper-left end is the point $F(0) = f(0,0)$, while the bottom end is $F(5) = f(5,5)$. Label all of the indicated points as polar values of the spline $F$. (Make several copies of Figure 1, so that you can do exploratory scribbling on some of the copies and still have a clean copy left to mark up and include as part of your answer to this problem.)

The spline $F$ has the unusual property that one of its joints—either $F(1)$, $F(2)$, $F(3)$, or $F(4)$—isn't a joint at all. The parabolic segment entering that joint and the parabolic segment leaving that joint happen to be adjacent pieces from the same parabola. Figure out which of the joints has this special property. By drawing additional points and lines as necessary, show that we can delete the knot corresponding to the non-joint from the knot sequence of $F$ and hence

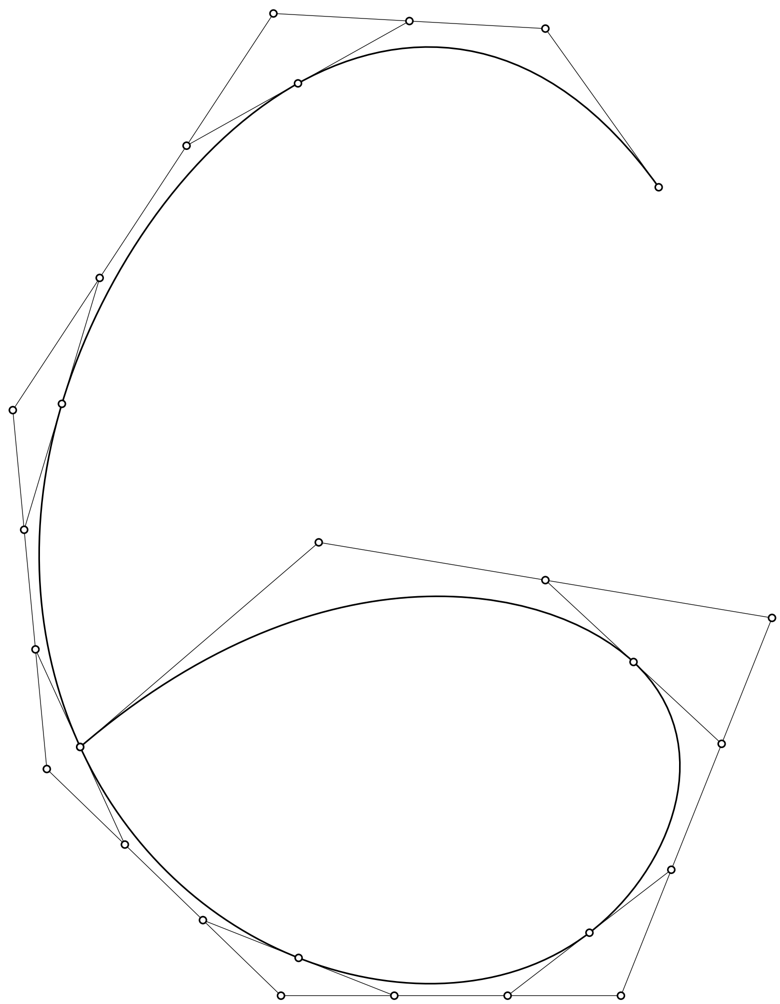

Figure 2: A cubic spline with one non-joint

interpret $F$ as a spline on a knot sequence of length 7. Be sure to label whatever new points you draw as polar values of $F$.

Figure 2 shows a cubic, polynomial spline curve $G$ on the knot sequence

$$(0,0,0,1,2,3,4,5,6,7,7,7),$$

of length 12. The spline $G$ looks something like the numeral six. The upper-right end is the point $G(0) = g(0,0,0)$, while the end at the T-joint is $G(7) = g(7,7,7)$ (which happens to be the same point as the joint $G(3) = g(3,3,3)$). Label all of the indicated points as polar values of the spline $G$.

The spline $G$ has the same unusual property as $F$: One of its joints isn't a joint at all. The cubic segment entering that joint and the cubic segment leaving that joint happen to be adjacent pieces from the same cubic polynomial curve. Figure out which of the joints of $G$ has this special property. By drawing additional points and lines, show that we can delete the knot corresponding to the non-joint from the knot sequence of $G$ and hence interpret $G$ as a spline on a knot sequence of length 11. Label whatever new points you draw as polar values of $G$. The new points that you draw should include all of the de Boor points corresponding to the shortened knot sequence.