Homework #2: More on tranformations and quaternions; Curves and their polar forms [65 points]
Due Date: Tuesday, 8 February 2005

Problem 1. [15 points]

Choose three points in the plane, label them $f(0,0)$, $f(0,3)$, and $f(3,3)$, and sketch the parabolic segment $F([0\ ..\ 3])$ that has those three points as its Bézier points and f as its polar form. Add points and lines to your sketch so as to construct $F(1)$ and $F(2)$, and label whatever new points you have added as polar values of F. Redraw your sketch, if necessary, to make it neat and clear. Considering the resulting figure, give two different recipes for constructing the point $f(1,2)$ from other polar values. Why do these two recipes give the same answer? What does this have to do with the medians of the triangle whose vertices are $f(0,1)$, $f(0,3)$, and $f(2,3)$?

Problem 2. [20 points]

Let $F([0\,..\,1])$ be the cubic segment in the plane whose four Bézier points are $f(0,0,0) := (0,0)$, $f(0,0,1) := (\lambda,\lambda)$, $f(0,1,1) := (1-\lambda,\lambda)$, and $f(1,1,1) := (1,0)$, where λ is some positive number. For each positive λ, classify the cubic F as either humpy, pointy, loopy, or parabolic. For which of the λ that make F humpy do the two flexes (points of inflection) on F correspond to times t in the interval $[0\ ..\ 1]$? For which of the λ that make F loopy do the two times t that correspond to F's self-intersection lie in the interval $[0\ ..\ 1]$?

Hints: Compute the X and Y coordinates of $F(t)$, say $X(t,\lambda)$ and $Y(t,\lambda)$. An inflection point, called a *flex*, is a point where the velocity and acceleration vectors are parallel. A self-intersection, called a *crunode*, is a single point that corresponds to two different times: $F(t_1) = F(t_2)$. A humpy cubic has two real, finite flexes, but no crunode. A loopy cubic has a crunode, but no real, finite flexes.

Problem 3. [15 points]

Figure 1 shows a quadratic, polynomial spline curve F on the knot sequence

$$(0,0,1,2,3,4,5,5),$$

of length 8. The spline F looks something like a question mark. The upper-left end is the point $F(0) = f(0,0)$, while the bottom end is $F(5) = f(5,5)$. Label all of the indicated points as polar values of the spline F. (Make several copies of Figure 1, so that you can do exploratory scribbling on some of the copies and still have a clean copy left to mark up and include as part of your answer to this problem.)

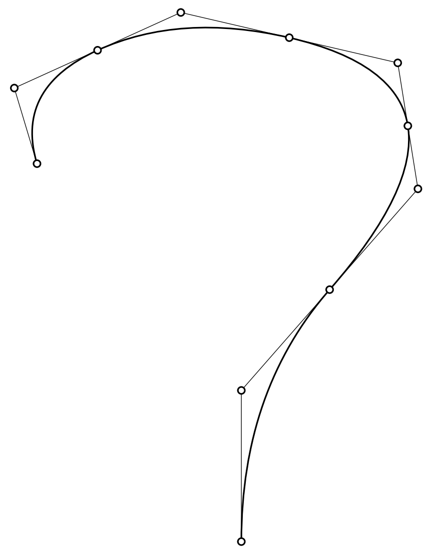

Figure 1: A quadratic spline with one non-joint

The spline F has the unusual property that one of its joints—either $F(1)$, $F(2)$, $F(3)$, or $F(4)$—isn't a joint at all. The parabolic segment entering that joint and the parabolic segment leaving that joint happen to be adjacent pieces from the same parabola. Figure out which of the joints has this special property. By drawing additional points and lines as necessary, show that we can delete the knot corresponding to the non-joint from the knot sequence of F and hence interpret F as a spline on a knot sequence of length 7. Be sure to label whatever new points you draw as polar values of F.

Figure 2 shows a cubic, polynomial spline curve G on the knot sequence

$$(0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 7, 7),$$

of length 12. The spline G looks something like the numeral six. The upper-right end is the point $G(0) = g(0, 0, 0)$, while the end at the T-joint is $G(7) = g(7, 7, 7)$ (which happens to be the same point as the joint $G(3) = g(3, 3, 3)$). Label all of the indicated points as polar values of the spline G.

The spline G has the same unusual property as F: One of its joints isn't a joint at all. The cubic segment entering that joint and the cubic segment leaving that joint happen to be adjacent pieces from the same cubic polynomial curve. Figure out which of the joints of G has this special property. By drawing additional points and lines, show that we can delete the knot corresponding to the non-joint from the knot sequence of G and hence interpret G as a spline on a knot sequence of length 11. Label whatever new points you draw as polar values of G. The new points that you draw should include all of the de Boor points corresponding to the shortened knot sequence.

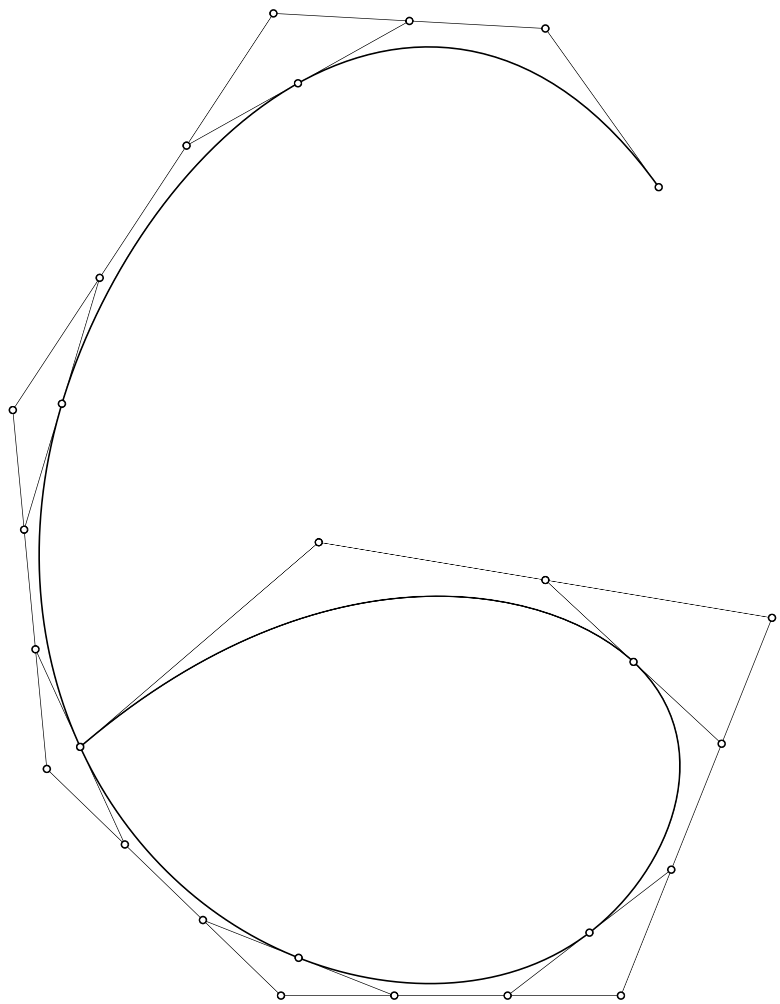

Figure 2: A cubic spline with one non-joint

Problem 4. [15 points]

Let $p := (1 + I)/\sqrt{2}$ and $q := (1 + J)/\sqrt{2}$ denote the unit-norm quaternions used in the example in the reader (page 108). Recall that the rotation $M(p)$ is a 90-degree rotation about the X axis, while $M(q)$ is a 90-degree rotation about the Y axis. In the reader, we composed the two rotations $M(p)$ and $M(q)$. Here, we instead investigate the rotation that lies halfway between $M(p)$ and $M(q)$.

The quaternion that lies halfway between p and q is simply

$$\frac{p+q}{2} = \frac{1}{\sqrt{2}} + \frac{I}{2\sqrt{2}} + \frac{J}{2\sqrt{2}}.$$

Calculate the norm $|(p+q)/2|$ of that quaternion, and note that it is not 1. Find a quaternion r that is a scalar multiple of $(p+q)/2$ and that has unit norm, $|r| = 1$, and calculate the rotation matrix $M(r)$. Around what axis does $M(r)$ rotate, and through what angle (say, to the nearest tenth of a degree)?

Find a globe and a piece of string. Say that Leo's Bistro is located at 45 degrees north latitude and 0 degrees longitude, near Bordeaux in France, and that Nikola's Bar and Grill is located at 45 degrees north latitude and 90 degrees west longitude, in the middle of Wisconsin. To the nearest degree, what is the latitude of an airplane that is halfway along a great-circle course from Leo's to Nikola's?

Briefly explain the connection between the two halves of this problem. In particular, which quaternions correspond to which points on the globe?