



Communication

The upper bound theorem for polytopes: an easy proof of its asymptotic version

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Abstract

Since at least half of the d edges incident to a vertex v of a simple d -polytope P either all point “up” or all point “down,” v must be the unique “bottom” or “top” vertex of a face of P of dimension at least $d/2$. Thus the number of P ’s vertices is at most twice the number of such high-dimensional faces, which is at most $\sum_{d/2 \leq i \leq d} \binom{n}{i} = O(n^{\lfloor d/2 \rfloor})$, if P has n facets.

This, in a nutshell, provides a proof of the asymptotic version of the famous upper bound theorem: A convex d -polytope with n facets (or, dually, with n vertices) has $O(n^{\lfloor d/2 \rfloor})$ faces, when d is constant.

1. Remarks and discussion

For a d -polytope P let $f_i(P)$ denote the number of i -faces of P . How large can $f_i(P)$ get, provided $f_{d-1}(P) = n$, i.e. P has n facets? In 1970 McMullen [5] proved the exact bound of

$$f_i(P) \leq \sum_{0 \leq j \leq d} \binom{j}{i} \binom{n-1-\max\{j, d-j\}}{\min\{j, d-j\}},$$

after Klee [4], [3, sec. 10.1] had proven it for sufficiently large values of n . Their proofs relied on the so-called Dehn–Sommerville relations and in McMullen’s case on the existence of shellings.

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The purpose of this note is to point out that if one is interested only in asymptotic bounds, the simple argument of the abstract essentially suffices. To flesh out the argument one only needs the following observations:

- (i) Among the d -polytopes with n facets the face count $f_i(P)$ is maximized for each i when P is a simple polytope, i.e. each i -face of P is the intersection of exactly $d - i$ facets (see [3, p. 80] for a proof in the dual setting).
- (ii) A “vertical” direction can be chosen so that no two vertices of P are at the same “height.”
- (iii) For each i edges emanating from a vertex of a simple d -polytope P there is a unique i -face of P containing those edges.
- (iv) Trivially, $f_i(P) \leq \binom{n}{d-i}$, and because of (iii) we have $f_i(P) \leq \binom{d}{i} f_0(P)$.

One can do without observation (i), if one is only interested in a bound for $f_0(P)$. However, since for non-simple polytopes observation (iii) and hence the second inequality in (iv) fails, this vertex bound does not carry over to bounds for faces of general dimension. For the vertex case observe that even for a non-simple d -polytope P , every vertex is the top or bottom vertex of a face of P of dimension at least $d/2$. This follows from the fact that if a k -polytope Q (the vertex figure of the vertex in question) is cut by a hyperplane H not containing any vertices of Q , then some face of dimension at least $\lfloor k/2 \rfloor$ is not cut by H . (Since Q is the union of the convex hulls of pairs of uncut faces on opposite sides of H , not having a large-dimensional uncut face would imply that Q does not have dimension k .)

Our argument shows that $f_0(P) \leq 2 \cdot \sum_{d/2 < i \leq d} f_i(P)$. Chazelle, Matoušek, Sharir, and Welzl [1] have pointed out that the argument can be somewhat refined to show that $f_0(P) \leq 2 f_{\lfloor d/2 \rfloor}(P)$ holds for odd d , and $f_0(P) \leq f_{d/2}(P)$ for even d .

Dey [2] has used a scheme similar to ours to show that any triangulation of n points in \mathbb{R}^d contains $O(n^{\lfloor d/2 \rfloor})$ simplices.

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